Week 5 Description
7.8: Improper Integrals
8.1: Arc Length

Improper integrals come in two flavors. One where the length of the path of integration is infinite such as

$\int_a^\infty f$ or $\int_{-\infty}^a f$ or $\int_{-\infty}^\infty f$
for example

1. $\displaystyle\int_1^\infty \frac{dx}{x^2}$
2. $\displaystyle\int_{-\infty}^0 xe^x dx$
3. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+1}$

All require taking limits. By definition

$$\int_a^\infty f = \lim_{t\to\infty}\int_a^t f$$

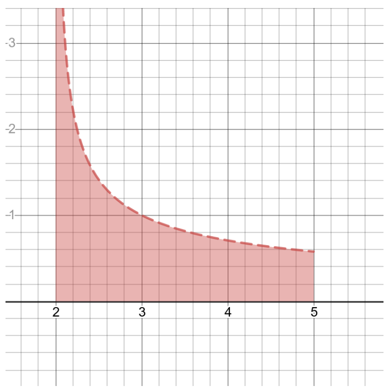
1. $\displaystyle\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} -\frac{1}{x}\bigg|_1^t = \lim_{t\to\infty} 1-\frac{1}{t} = 1$
2. $\displaystyle\int_{-\infty}^0 xe^x dx = \lim_{t\to-\infty}\int_t^0 xe^x dx = \lim_{t\to-\infty} -te^t - 1 + e^t \text{ (why?) } = -1$
3. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+1}$ must be broken in to two integrals each improper $\displaystyle\int_{-\infty}^0 \frac{dx}{x^2+1} + \int_0^\infty \frac{dx}{x^2+1}$
   The first is $\displaystyle\lim_{t\to-\infty} -\tan^{-1}(t) = \frac{\pi}{2}$ (why?) and the second is the same.

Because we are taking a limit in each case, it is always possible that the limit does not exist. If it does exist, we say the integral **converges** and if not we say it **diverges**.
Worlds most famous example of divergence would be

$$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\log(t) = \infty$$

Second flavor is where f is discontinuous at some number in the interval of integration such as

1. $\int_2^5 \frac{dx}{\sqrt{x-2}}$ which improper because $f(x) = \frac{1}{\sqrt{x-2}}$ is not continuous at 2



In this case we take

$$\lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} 2\sqrt{x-2}\Big|_t^5 = 2\sqrt{3}$$

2. The following must be broken in to two integrals since $f(x) = \frac{1}{x-1}$ is discontinuous at $x = 1$

$$\int_{-2}^2 \frac{dx}{x-1} = \int_{-2}^1 \frac{dx}{x-1} + \int_1^2 \frac{dx}{x-1}$$

It is an easy check that they do not exist because

$$\lim_{t\to 1} \log(t-1) = -\infty$$

The derivation of the Arc Length formula that says the length of the curve of a continuous function f for $a \le x \le b$ is

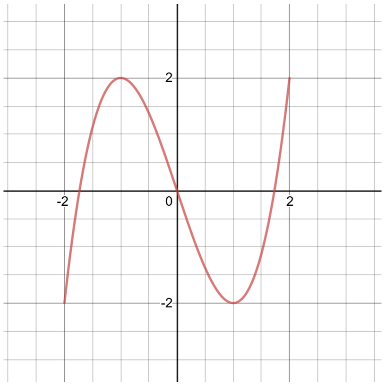
$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

is in the book.

The most remarkable thing about the exercises is that they make up examples where, when you take the derivative of a function, square it, add one, then take the square root, you get something you can actually find the anti-derivative of. We don't care about such examples because using numeric integration via wolfram we can find almost any arc length.

Even a seemingly simple example like

1. Find the arc length of $y = x^3 - 3x$ for $-2 \leq x \leq 2$



would be almost impossible but we can do it easily.

Here $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3$ making the arc length

$$L = \int_{-2}^{2} \sqrt{1 + (3x^2 - 3)^2} dx \approx 13.0371$$

The answer being found **here**