Week 13 Description
10.3 Polar Coordinates
10.4 Areas and lengths
Just like last week this is a job for wolfram.

The idea is that instead of using Cartesian coordinates$(x, y)$ you use $(r, \theta)$ where $r$ is the distance from the origin and $\theta$ an angle from the polar axis (see figure 1, page 658) Here is a pretty good **Basic Introduction**
Another good introduction is **Here**

It is easy to switch from polar to Cartesian via

$$x = r\cos(\theta), y = r\sin(\theta)$$

To go the other way, use

$$x^2 + y^2 = r^2, \cos(\theta) = \frac{x}{r}, \sin(\theta) = \frac{y}{r}$$

There are several questions on graphs, such as

1. $r = 1 + \sin(\theta)$ which you should try by hand, but it is a **Cardioid**

   The second part on derivatives takes more work.

   If $r$ is a function of $\theta$ i.e. $r = r(\theta)$ then since $x = r\cos(\theta)$ we have

$$\frac{dx}{d\theta} = r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)$$

   by the product rule. Similarly

$$\frac{dy}{d\theta} = r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)$$

   And therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

   I have never memorized this, you use the product rule as needed. The book makes it look harder than it is because it uses the formula. You don't need to. If for example

2. If $r = 1 + \cos(\theta)$ **Cardioid**

   Then $y = r\sin(\theta) = (1+\cos(\theta))\sin(\theta) = \sin(\theta) + \cos(\theta)\sin(\theta)$ and

   $$\frac{dy}{d\theta} = \cos(\theta) + \cos^2(\theta) - \sin^2(\theta)$$

   by the product rule

   The tangent will be horizontal when this is equal to 0, at $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$ and the corresponding points are $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0, \pi)$, $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$

   10.4 Areas and lengths

   The area of a polar region is given by

   $$A = \int_a^b \frac{1}{2} r^2 d\theta$$

   The integral is not an integral along the x axis, it is integration by sweeping across an angle. A very nice introduction and illustration is here: **Abby Brown**

   If you do the integrals by hand, they will repeatedly use

   $$\int \sin^2(\theta) d\theta = \frac{\theta}{2} - \frac{\sin(2\theta)}{4}$$

   $$\int \cos^2(\theta) d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4}$$

   If you do them using wolfram it requires almost no work, for example

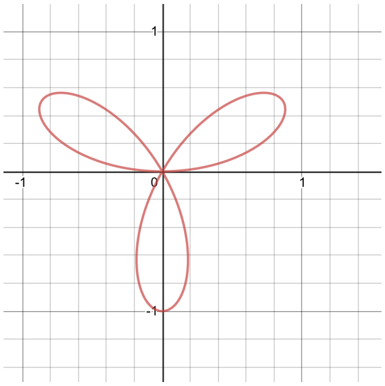
3. The area enclosed by $r = 2 + \sin(\theta)$ is

   $$\frac{1}{2}\int_0^{2\pi} (2+\sin(\theta))^2 d\theta = \frac{9\pi}{2}$$

   **Wolf**

   Since we are cheating, the only hard part may be finding the limits of integration.

4. Find the area enclosed by one loop of $r = \sin(3\theta)$




If it is not obvious, the limits can be found by setting $\sin(3\theta) = 0$ and solving twice:

$$\sin(3\theta) = 0 \to 3\theta = 0 \to \theta = 0$$

$$\sin(3\theta) = 0 \to 3\theta = \pi \to \theta = \frac{\pi}{3}$$

Once we have the limits it is routine: **Wolf**

Finally we get to the arc length formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Only one question for this, see if you can set up the integral correctly.

5. Find the arc length of the cardioid $r = 1 + \cos(\theta)$

$$\int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta d\theta} = \int_0^{2\pi} \sqrt{2 + 2\cos\theta} d\theta = 8$$

**Wolf**