A baby proof of the fundamental theorem of calculus for the specific case of

$$\int_1^x t^2 dt$$

Theorem says that if f is continuous on $[a, b]$ and $a < x < b$ then the derivative of

$$G(x) = \int_1^x t^2 dt$$

is $f(x)$ i.e. $G'(x) = f(x)$
In this case we should be able to prove that the derivative of

$$\int_1^x t^2 dt$$

is $f(x) = x^2$
First we need to remember what the definition of the derivative is:

$$G'(x) = \lim_{h \to 0} \frac{G(x+h) - G(x)}{h}$$

Second we need to figure out how to apply that general definition in the specific case where

$$G(x) = \int_1^x t^2 dt$$

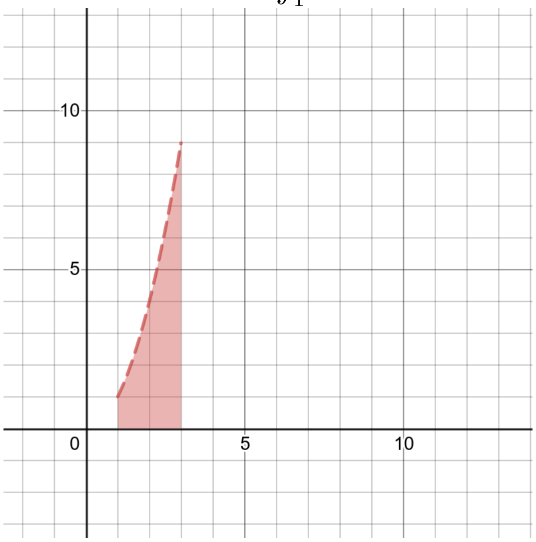
It is confusing but important to remember that the variable is x not t and $G(x)$ is a function of x so for example

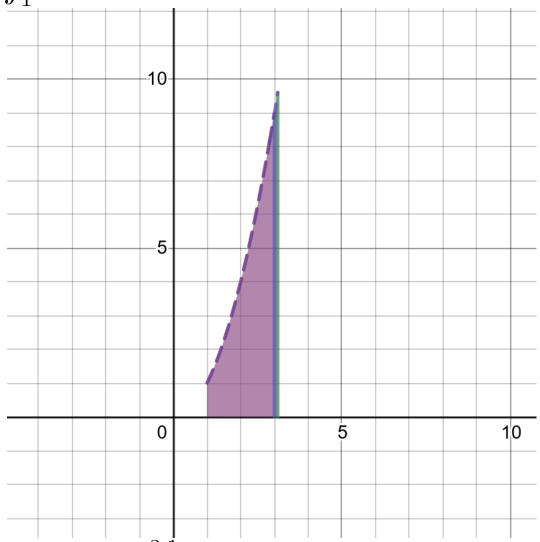
$$G(5) = \int_1^5 t^2 dt, G(\pi) = \int_1^\pi t^2 dt, G(\pi + \sqrt{2}) = \int_1^{\pi+\sqrt{2}} t^2 dt$$

and so

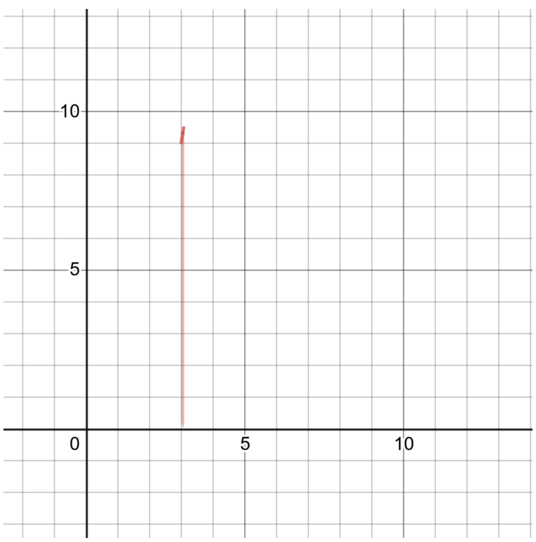
$$G(x+h) = \int_1^{x+h} t^2 dt$$

Here is a picture of $\int_1^3 t^2dt$ on the left $\int_1^{3.1} t^2dt$





Subtracting the first one from the second gives $\int_3^{3.1} t^2dt$



With these pictures in mind we can see that

$$G(x+h) - G(x) = \int_1^{x+h} t^2dt - \int_1^x t^2dt = \int_x^{x+h} t^2dt$$

On the interval $[x, x+h]$ for $h > 0$ the largest this integral can be is $h(x+h)^2$ and the smallest it can be is hx^2 since the length of the path is h and t^2 is increasing. so

$$hx^2 \leq \int_x^{x+h} t^2dt \leq h(x+h^2)$$

Dividing by h we get

$$\frac{G(x+h)-G(x)}{h} = \frac{1}{h}\int_x^{x+h} t^2 dt$$

and so

$$x^2 \leq \frac{G(x+h)-G(x)}{h} \leq (x+h)^2$$

Taking the limit as $h \to 0$ the left hand limit is x^2 as there is no h in it, the limit on the right is x^2 as well, (replace h by 0), and the middle is the definition of $G'(x)$ i.e.

$$x^2 \leq G'(x) \leq x^2$$

making

$$G'(x) = x^2$$

The general proof is very similar, replacing t^2 by $f(t)$ and justifying maximum and minimum value of the integral by the extreme value theorem, and justifying the last limit on the left equal to the limit on the right by the continuity of f

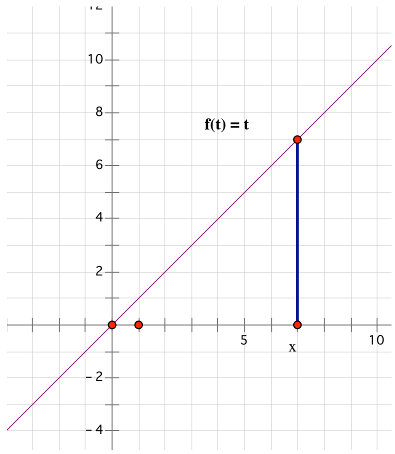
The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus shows that differentiation and Integration are inverse processes.

Consider the function $f(t) = t$. For any value of $x > 0$, I can calculate the definite integral

$$\int_0^x f(t)dt = \int_0^x tdt.$$

by finding the area under the curve:



This gives us a formula for $\int_0^x f(t)dt$ in terms of x, in fact we see that it is a function of x:

$$F(x) = \int_0^x tdt =$$

What is $F'(x)$?

This is an example of a general phenomenon for continuous functions:

The Fundamental Theorem of Calculus, Part 1 : If f is a **continuous function** on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt, \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$ or

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

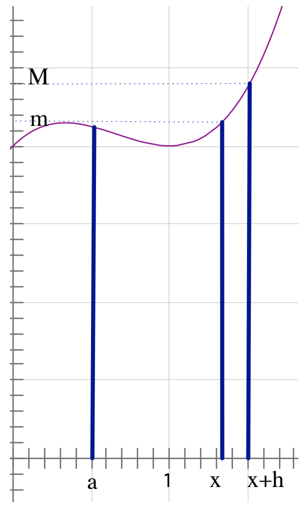
Note This tells us that $g(x)$ **is an antiderivative for** $f(x)$**.**

Proof We know that

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$

First we will focus on putting the quotient on the right hand side into a form for which we can calculate the limit. Using the definition of the function $g(x)$, we get

$$\frac{g(x+h)-g(x)}{h} = \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$



If $f(x) > 0$ the integral $\int_x^{x+h} f(t)dt$ is that area between the curve $y = f(t)$ and the t-axis, over the interval from $t = x$ and $t = x + h$. Since f is continuous on the interval $[x, x+h]$, we can use the Extreme Value Theorem to show that f achieves a maximum, M, and a minimum, m, on that interval. That is, for all values of t in the interval $[x, x+h]$,

$$m \le f(t) \le M$$

and by the laws of definite integrals, we have

$$m(x+h-x) \le \int_x^{x+h} f(t)dt \le M(x+h-x) \quad \text{or} \quad mh \le \int_x^{x+h} f(t)dt \le Mh.$$

Dividing across by h, we get

$$m \le \frac{1}{h}\int_x^{x+h} f(t)dt \le M.$$

The minimum and maximum are not necessarily at the endpoints of the interval as shown in the picture above, they may be some where in the interior. However the Extreme Value Theorem (which applies because the function is continuous) guarantees that there is a number c_1 in the interval with $f(c_1) = m \le f(t)$ for all $t \in [x, x+h]$ and there is a number $c_2 \in [x, x+h]$ for which $f(c_2) = M \ge f(t)$ for all $t \in [x, x+h]$. So this gives us

$$f(c_1) \le \frac{1}{h}\int_x^{x+h} f(t)dt \le f(c_2)$$

where $c_1, c_2 \in [x, x+h]$.

Now taking limits, we get

$$\lim_{h\to 0} f(c_1) \ \le\ \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt \ \le\ \lim_{h\to 0} f(c_2)$$

As $h \to 0$, $c_1 \to x$ and $c_2 \to x$, because the width of the interval is going to 0. Because $f(t)$ is continuous

$$\lim_{h \to 0} f(c_1) = f(x) = \lim_{h \to 0} f(c_2)$$

and

$$f(x) \;\leq\; \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt \;\leq\; f(x) \quad \text{and} \quad \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x)$$

This proves that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x).$$

Example Find the derivative of the functions listed below:

$$g(x) = \int_1^x \sqrt{9 + t^2}dt, \qquad h(x) = \int_5^x \frac{1}{\sqrt{1 + \cos^2 t}}dt$$

Note A careful look at the proof of the above theorem shows that it also applies to the situation where $a \leq x \leq b$:

> If f is a **continuous function** on $[a, b]$, then the function g defined by
>
> $$g(x) = \int_b^x f(t)dt, \qquad a \leq x \leq b$$
>
> is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$ or
>
> $$\frac{d}{dx} \int_b^x f(t)dt = f(x).$$

This implies that

$$\boxed{\frac{d}{dx} \int_x^b f(t)dt = \frac{d}{dx}\left(- \int_b^x f(t)dt \right) = -f(x).}$$

Example Find the derivative of the function:

$$F(x) = \int_x^1 \frac{1}{3 + \cos u}du$$

We can also use **the chain rule** with the Fundamental Theorem of Calculus:
Example Find the derivative of the following function:

$$G(x) = \int_1^{x^2} \frac{1}{3 + \cos t} dt$$

The Fundamental Theorem of Calculus, Part II If f is **continuous** on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (\text{ notation} F(b) - F(a) = F(x)\Big|_a^b \ )$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Proof Let $g(x) = \int_a^x f(t)dt$, then from part 1, we know that $g(x)$ is an antiderivative of f. Hence if $F(x)$ is another antiderivative for f, we have

$$F(x) = g(x) + C$$

for some constant C and $a < x < b$. Since F and g are continuous, we see by taking limits that $F(a) = g(a) + C$ and $F(b) = g(b) + C$.

Now

$$g(a) = \int_a^a f(t)dt = 0 \text{ and } g(b) = \int_a^b f(t)dt$$

Therefore

$$F(b) - F(a) = (g(b) + C) - (g(a) + C) = g(b) - g(a) = \int_a^b f(t)dt.$$

This makes the calculation of integrals much easier for any function for which we can find an antiderivative.

Example Evaluate the following integrals:

$$\int_{-1}^1 x^2 dx, \qquad \int_1^3 \frac{1}{x^2} dx \qquad \int_0^{\frac{\pi}{2}} \cos x dx \qquad \int_0^{\frac{\pi}{4}} \sqrt{x} + 2\sec^2 x dx$$

Example Why is the above method not applicable to

$$\int_{-1}^1 \frac{1}{t^2} dt?$$

Cauchy and The Rigorous Development of Calculus

The Approaches of Newton and Leibniz to Calculus

From foundations provided by earlier mathematicians such as Barrow during the first part of the 17th century, Sir Isaac Newton (1642--1727) mastered concepts of tangent and quadrature (definite integration).
His interpretations were based on physical models of time, motion, and velocity.

In a letter to Gottfried Wilhelm Leibniz (1646--1716), Newton stated the two most basic problems of calculus
were

"*1. Given the length of the space continuously* [i.e., at every instant of time]*, to find the speed of motion* [i.e., the derivative] *at any time proposed.*
2. Given the speed of motion continuously, to find the length of the space [i.e., the integral or the antiderivative] described at any time proposed."

This indicates his understanding (but not proof) of the Fundamental Theorem of Calculus.

Instead of using derivatives, Newton referred to ***fluxions*** of variables, denoted by x, and instead of antiderivatives, he used what he called ***fluents***. Newton considered lines as generated by points in motion, planes as generated by lines in motion and bodies as generated by planes in motion, and he called these fluents. He used the term fluxions to refer to the velocity of these fluents.

Newton began thinking of the traditional geometric problems of calculus in algebraic terms. Newton's three calculus monographs were circulated to his colleagues of the Royal Society, but they were not published until much later, after his death.

Leibniz's ideas about integrals, derivatives, and calculus in general were derived from close analogies with
finite sums and differences. Leibniz also formulated an early statement of the Fundamental Theorem of
Calculus, and then later in a 1693 paper Leibniz stated, "the general problem of quadratures can be reduced to the finding of a curve that has a given law of tangency.

A ugly dispute between Leibniz and Newton, fueled by their followers ensued over credit for the development of these ideas. Most English mathematicians continued to Newton's fluxions and fluents, avoiding avoided Leibniz's superior notations until the early 1800's.

Both Newton and Leibniz developed calculus with an intuitive approach. Formal proofs came with later mathematicians, primarily Cauchy.

Augustin-Louis Cauchy (1789--1857)

(From the The MacTutor History of Mathematics Archive)

The rigorous development of the calculus is credited to Augustin Louis Cauchy (1789--1857). The modern proof of the Fundamental Theorem of Calculus was written in his *Lessons Given at the École Royale Polytechnique on the Infinitesimal Calculus* in 1823. Cauchy's proof finally rigorously and elegantly united the two major branches of calculus (differential and integral) into one structure.

Cauchy was born in Paris the year the French revolution began. Laplace was his neighbor, and Lagrange was an a friend and supporter. He was admitted to the École polytechnique in 1805 to study engineering at the age of 16. Cauchy had already read Laplace's Mécanique céleste and Lagrange's Traité des functions analytiques.

In 1816 he won a contest of the French Academy on the propagation of waves on the surface of a liquid.
In the same year when Monge and Carnot were expelled from the Académie des sciences, Cauchy was appointed as a replacement member. Eventually, Cauchy was appointed a full professor at the École polytechnique. His classic works Cours d'analyse (Course on Analysis,

1821) and Résumé des leçons ... sur le calcul infinitésimal (1823) contain his contributions to the rigorous development calculus. From 1831 to 1833, while in excile from France due to political unrest, he taught at the University of Turin in Switzerland, and subsequently accepted a professorship of celestial mechanics at Sorbonne. Cauchy was a highly prolific writer, publishing a total of 789 works.

Rigorous Calculus Begins with Limits

The major ideas of calculus – derivative, continuity, integral, convergence/divergence of sequences and series– are defined in terms of limits.

Limit is therefore the most fundamental concept of calculus . This concept of limit distinguishes calculus from other branches of mathematics such as algebra, geometry, number theory, and logic.

The currently used definition of limit is less than 150 years old. Before this time, the notions of limit were vague and confusing intuitions -- only infrequently used correctly. In fact, in much of his work on calculus, Isaac Newton failed to acknowledge the fundamental role of the limit.

In the beginning of Book I of the *Principia Mathematica*, Newton provides a formulation of the definition of limit :

"*Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and*
before the end of that time approach nearer to each other than by any given difference, become ultimately equal."

Concern about the lack of rigorous foundations for calculus grew during the late years of the 18th century
At the beginning of the 18th century, the ideas about limits were certainly confusing.

In 1821, Cauchy was searching for a rigorous development of calculus to present to his engineering students at the École polytechnique in Paris. He started calculus course from scratch; beginning with a modern definition of the limit. His class notes were essentially textbooks, the first one called Cours d'analyse (Course of Analysis). In his writings, Cauchy used limits as the basis for rigorous definitions of continuity and convergence, the derivative and the integral. He gave as his ***definition of limit***:

" *When the values successively attributed to a particular variable approach indefinitely a fixed value so as to differ from it by as little as one wishes, this latter value is called the limit of the others.* "

Karl Weierstrass (1815--1897), a professor of mathematics at the University of Berlin, restated Cauchy's original definition of the limit in strict arithmetical terms, using only absolute values and inequalities, giving us the epsilon-delta definition we use today.

Cauchy's definition of the ***derivative*** was given as:

"*The limit of [f(x + i) – f(x)] / i as i approaches 0. The form of the function which serves* *as the limit of the ratio [f(x + i) – f(x)] / i will depend on the form of the proposed function y = f(x). In order to indicate this dependence, one gives the new function the name of derived function.* "

Cauchy went on to find derivatives of all the elementary functions and to give the chain rule. He also applied the Mean Value Theorem for derivatives in the proof of a number of basic calculus results such as the first derivative criteria for increasing and decreasing functions.

Cauchy defined the ***integral*** of any continuous function on the interval [*a,b*] to be the limit of the sums of areas of thin rectangles. He attempted to prove that this limit existed for all functions continuous on the given interval. His attempted proof used the Intermediate Value Theorem, but contained some logical gaps.

Cauchy proved the Mean Value Theorem for Integrals and used it to prove the Fundamental Theorem of Calculus for continuous functions, giving the form of the proof used today's calculus texts.

Cauchy the first to define fully the ideas of convergence and absolute convergence of infinite series, including the development of the ratio and root tests for convergence of series.

He was also the first to develop a systematic theory for complex numbers and to develop the Fourier transform for differential equations.