Week 4 notes

First batch of problems about limits at ∞

1. Rational functions

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

where P, Q are polynomials.

(i) If the degree of $Q$is greater then the degree of P the limit is zero.

(ii) If the degree of P is bigger than the degree of Q there is no limit (or maybe we say ∞

(iii) If the degrees are equal, the limit is the ratio of the leading coefficients.

Forget about all the stuff they write in the book about dividing by the highest degree. This is the same as finding the horizontal asymptotes in math 161.

(a)

$$\lim_{x\to\infty} \frac{x^2 - x + 2}{3x^2 + 10x} = \frac{1}{3}$$

case (iii)

(b)

$$\lim_{x\to\infty} \frac{1}{x-5} = 0$$

case (i)

(c)

$$\lim_{x\to\infty} \frac{x^2 + 2x}{x-5} = \infty$$

case (ii)

(d)

$$\lim_{x\to\infty} \frac{1-2x}{5+x} = -2$$

case (iii) again

2.

$$\lim_{x\to\infty} \frac{\sqrt{10x^2+3x}}{2x+3}$$

not a rational function because the numerator is not a polynomial. No matter, ignore everything but the terms of highest degree top and bottom and think

$$\lim_{x\to\infty} \frac{\sqrt{10x^2+3x}}{2x+3} = \lim_{x\to\infty} \frac{\sqrt{10x^2}}{2x} = \lim_{x\to\infty} \frac{\sqrt{10}x}{2x} = \frac{\sqrt{10}}{2}$$

3.

$$\lim_{x\to-\infty} x+\sqrt{x^2+3x}$$

$$x+\sqrt{x^2+3x}\times\frac{x-\sqrt{x^2+3x}}{x-\sqrt{x^2+3x}}=\frac{-3x}{x-\sqrt{x^2+3x}}=\frac{3x}{\sqrt{x^2+3x}-x}$$

Replace x by $-x$ and proceed as in the previous case.

$$\lim_{x\to\infty}\frac{-3x}{\sqrt{x^2}+x}=\lim_{x\to\infty}\frac{-3x}{2x}=-\frac{3}{2}$$

4. Find the slope of the tangent line to $f(x)=x^2+4x-1$ at the point $(2,11)$

   Method one

$$m=\lim_{x\to2}\frac{x^2+4x-1-11}{x-2}=\lim_{x\to2}\frac{x^2+4x-12}{x-2}=\lim_{x\to2}\frac{(x-2)(x+6)}{x-2}=\lim_{x\to2}x+6=8$$

   Method two

$$m=\lim_{h\to0}\frac{(2+h)^2+4(2+h)-1-11}{h}$$

$$(2+h)^2+4(2+h)-1-11=4+4h+h^2+8+4h-12=8h+h^2$$

   Divide by h get $8+h$ and

$$\lim_{h\to0}8+h=8$$

5. Find the derivative of $f(x)=x^2+4x-1$ at the point $(a,f(a))$

   Using method 2,

$$f'(a)=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$$

$$=\lim_{h\to0}\frac{(a+h)^2+4(a+h)-1-(a^2+4a-1)}{h}$$

$$=\lim_{h\to0}\frac{a^2+2ah+h^2+4a+4h-1-a^2-4a+1}{h}$$

$$=\lim_{h\to0}\frac{2ah+4h+h^2}{h}=\lim_{h\to0}2a+4+h=2a+4$$

6. Find the slope of the line tangent to $f(x) = \sqrt{x}$ at the point $(9, 3)$

NO thank you! Why don't we just find it at the point $(a, \sqrt{a})$ and then we can plug in 9 or 16 or even 5.

$$\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

usual trick multiply by the conjugate

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}$$

and

$$\lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

Again using second method

$$\lim_{h \to 0} \frac{\sqrt{a + h} - \sqrt{a}}{h}$$

$$\frac{\sqrt{a + h} - \sqrt{a}}{h} \times \frac{\sqrt{a + h} + \sqrt{a}}{\sqrt{a + h} + \sqrt{a}} = \frac{a + h - a}{h\left(\sqrt{a + h} + \sqrt{a}\right)} = \frac{1}{\sqrt{a + h} + \sqrt{x}}$$

and

$$\lim_{h \to 0} \frac{1}{\sqrt{a + h} + \sqrt{x}} = \frac{1}{2\sqrt{a}}$$

therefore at $(9, 3)$ the slope is $\frac{1}{6}$ and at $(16, 4)$ it is $\frac{1}{8}$ and at $(5, \sqrt{5})$ it is $\frac{1}{2\sqrt{5}}$