Week 7 Description
11.1 Sequences
Although this is just one section with only 10 pages not counting exercises, there is a lot of information here.
We can start easily with a sequence of numbers as a list

$$a_1, a_2, a_3, ...a_k...$$

where the subscripts just mean "first number', "second number', "kth number" etc
For our purposes it may be convenient to think of a sequence as a function f whose domain is $\mathbb{N}$ i.e. $f(n) = a_n$ but we will seldom write it this way.
Some sequences are given by a formula, just like functions. So for example you could have

$$a_n = \frac{n}{n+1} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ... \right\}$$

Or the sequence may be given by a **recursion**, for example

$$a_1 = 1, a_n = 3 - \frac{1}{a_{n-1}}$$

To see how this works, first compute a_2

$$a_2 = 3 - \frac{1}{a_{2-1}} = 3 - \frac{1}{a_1} = 3 - \frac{1}{1} = 3 - 1 = 2$$

Moving a little quicker, for a_3 we get

$$a_3 = 3 - \frac{1}{2} = \frac{5}{2}$$

and

$$a_4 = 3 - \frac{2}{5} = \frac{13}{5}, a_5 = 3 - \frac{5}{13} = \frac{34}{13}$$

Making the sequence

$$\left\{ 1, 2, \frac{5}{2}, \frac{13}{5}, \frac{34}{13} ... \right\}$$

The mother of all recursions is the Fibonacci series given by

$$f_1 = f_2 = 1, f_n = f_{n-1} + f_{n-2}$$

i.e. add the two previous numbers to get the next one

$$\{1, 1, 2, 3, 5, 8, 13, 21, ...\}$$

Another way to get a sequence of numbers is to take "partial sums"

$$a_1 = \frac{1}{2}, a_2 = \frac{1}{2} + \frac{1}{2^2}, a_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}, \ldots$$

We could write this as the recursion

$$a_1 = \frac{1}{2}, a_n = a_{n-1} + \frac{1}{2^n}$$

Or even more succinctly in sigma notation as

$$a_n = \sum_{k=1}^{n} \frac{1}{2^k}$$

There is an even easier way to write it which you can do for yourself by actually computing the first few terms and see the pattern.
Then come a bunch of definitions which should be memorized

Definition A sequence $\{a_n\}$ has a limit L written $\lim_{n\to\infty} a_n = L$ means for every $\epsilon > 0$ there is a corresponding natural number N such that if $n > N$ then $|a_n - L| < \epsilon$

Don't be freaked out by the formality, that means we can make the numbers in the sequence as close to L as we like by taking n large enough.

As for computing a limit, if we have a formula we can think of a_n as $f(n)$ and the limit as $n \to \infty$ would be a synonym for the horizontal asymptote of the function.

As with the terminology of improper integrals, if the limit exists we say it **converges** otherwise **diverges**

A "geometric" sequence $a_n = cr^{n-1}$ is a sequence in which one term is multiplied by by a fixed number (common ratio) to get the next number.
The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$

It is instructive to put $r = -1$, compute $(-1)^n$ and see why it does not converge. There is a lot more to sequences, including more definitions.

Definition: A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$ that is, $a_1 < a_2 < a_3 < \ldots$.
It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$
A sequence is **monotonic** if it is either increasing or decreasing.

Definition A sequence (a_n) is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

It is bounded below if there is a number m such that

$$m \leq a_n$$

for all $n \geq 1$

If it is bounded above and below, then $\{a_n\}$ is a **bounded** sequence.
Monotone Convergence Theorem Every bounded monotonic sequence converges.

There is a proof in the book. It casually drops the "Completeness Axiom" on you as if you have heard of it before and it is no big deal. Actually it is a big deal. If you want, check out the **axioms for real numbers** (you might want to look up "axiom" first) and see how mundane most of them are. Then look up why it is called the "Completeness" Axiom

One fact good to know is that if $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} a_{n-1} = L$ as well. You can work from the definition, but the sequence is the sequence.

For example if $\lim_{n\to\infty} a_n = \frac{2}{3}$ Then $\lim_{n\to\infty} 3 + \frac{1}{a_{n-1}} = 3 + \frac{3}{2}$

Two examples follow

1. Find the limit of the sequence given by the recursion

$$a_1 = 1, a_n = \frac{1}{4 - a_{n-1}}$$

The fact that this is decreasing bounded below by 0 insures that is has a limit, but proving it is decreasing requires induction. If you know mathematical induction, try it.

Using the fact that if we call the limit L then both $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_{n-1} = L$ we need only solve

$$L = \frac{1}{4 - L}$$

a fairly easy quadratic which gives

$$L = 2 \pm \sqrt{3}$$

but there is only one limit. Writing out the first few terms shows $a_n < 2$ making the answer

$$\lim_{n\to\infty} a_n = 2 - \sqrt{3}$$

2. Let

$$a_1 = a_2 = 1, a_n = 2a_{n-1} + 3a_{n-2}$$

First few terms are

$$1, 1, 5, 13, 41, 121, ...$$

an unbounded increasing sequence.

Put $b_n = \dfrac{a_n}{a_{n-1}}$ and find $\lim_{n\to\infty} b_n$ The first few terms are

$$1, 5, \frac{13}{5}, \frac{41}{13}, \frac{121}{41}, ...$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \frac{2a_{n-1} + 3a_{n+2}}{a_{n-1}} = 2 + 3\frac{a_{n-2}}{a_{n-1}} = 2 + \frac{3}{b_{n-1}}$$

You should supply reasons for each equal sign, especially the last one. But now we are almost done as before. If we call the limit L we solve

$$L = 2 + \frac{3}{L}$$

or

$$L^2 - 2L - 3 = 0 \iff (L-3)(L+1) = 0$$

and so $L = 3$