Week 6 Description
8.2 Area of Surface of Revolution
Appendix G and Spivak: The log and exponentiation

If f is positive and has a continuous derivative, the surface area of the surface obtained by rotating the curve $y = f(x)$ for $a \leq x \leq b$ about the x -axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx$$

Note that if you put $ds = \sqrt{1 + [f'(x)]^2}dx$ you can write the formula in the abbreviated form

$$S = \int 2\pi y ds$$

or if about the y -axis

$$S = \int 2\pi x ds$$

Like the problems with arc length, we don't care if the integrals look impossible, we can go ahead and use wolfram to get a numeric answer.

1. Find the surface area of $y = \sin(x)$ for $0 \leq x \leq \pi$ rotated about the x axis.

   Since $\sin'(x) = \cos(x)$ we get

   $$\int_0^\pi 2\pi \sin(x)\sqrt{1 + \cos^2(x)}dx$$

   This integral can be done using the substitution $u = \cos(x)$ then $u = \tan(\theta)$ a big mess, but it is easy with **wolfram** You even get a nice picture.

2. Find the surface area of $y = \sin(x)$ for $0 \leq x \leq \pi$ rotated about the y axis.

   Note that we are not going to solve for x, and also that since we have y as a function of x we are still going to use $ds = \sqrt{1 + \cos^2(x)}dx$, no dy involved. The only thing that changes is that instead of

   $$\int_0^\pi 2\pi \sin(x)\sqrt{1 + \cos^2(x)}dx$$

   we use

   $$\int_0^\pi 2\pi x\sqrt{1 + \cos^2(x)}dx$$

3. Rotate $y = \log(1+x), 0 \le x \le 1$ about the y axis. Since it is about the y axis we would use

$$S = \int 2\pi x ds$$

In this example $f'(x)^2 = \dfrac{1}{(x+1)^2}$ and the integral is

$$2\pi \int_0^1 x\sqrt{1 + \frac{1}{(x+1)^2}}dx \approx 3.695$$

wolfram

4. Rotate $x = y + y^3 + y^4, 0 \le y \le 1$ about the x axis.

Here we have x as a function of y (you cannot solve for y) so use $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy$ and solve

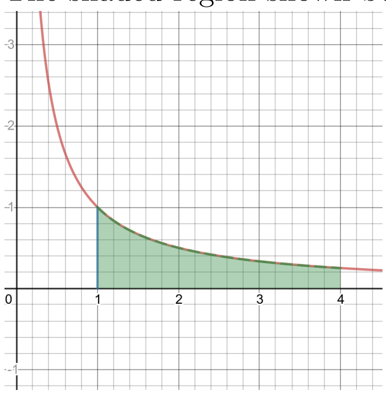
$$2\pi \int_0^1 y\sqrt{1 + (1 + 3y^2 + 4y^3)^2}dy$$

wolfram

Appendix G, and especially the material taken from Micheal Spivak Calculus is going to change the way we look at logs and exponents. Instead of defining the log as the inverse of an exponential we are going to use the definition

$$\log(x) := \int_1^x \frac{1}{t} dt, x > 0$$

The shaded region shown below would represent $\log(4)$



Working from the definition of the log as in integral we will derive all the familiar properties of the log that you already know

$$\log(xy) = \log(x)+\log(y), \log\left(\frac{x}{y}\right) = \log(x)-\log(y), \log(x) < 0 \text{ if } 0 < x < 1, \lim_{x\to\infty} \log(x) = \infty$$

And so on. We also see instantly that

$$\log(1) = \int_1^1 \frac{dt}{t} = 0$$

After that, the exponential function $e^x = \exp(x)$ will be defined as the inverse of the log. Then, for example, we will know that $e^0 = 1$ because $\log(1) = 0$ and not because "it is a rule"

The next step will be to define any exponential

$$b^x := e^{x \log(b)}$$

For example

1. $2^\pi = e^{\pi \log(2)}$
2. $4^{-1} = e^{-1 \log(4)}$
3. $\sin(x)^x = e^{x \log(\sin(x))}$

The first one gives meaning to raising a number to an irrational number.
The second show that $4^{-1} = \frac{1}{4}$ since

$$4^{-1} = e^{-\log(4)} = e^{\log\left(\frac{1}{4}\right)} = \frac{1}{4}$$

You should supply the reasons for each equal sign.
The last allows you to compute

$$\lim_{x \to 0} \sin(x)^x = \lim_{x \to 0} e^{x \log(\sin(x))}$$

by using L'Hôpital's rule to find $\lim_{x \to 0} x \log(\sin(x)) = 0$ and concluding that the limit is $e^0 = 1$
We also easily get the all important change of base formula that says the log of any base can be written in terms of the log defined as an integral.

$$x = b^{\log_b(x)} = e^{\log_b(x) \log(b)}$$

the first equal sign by definition of $\log_b$ and the second by the definition of the exponent. Taking the log of both sides gives

$$\log(x) = \log_b(x) \log(b) \iff \log_b(x) = \frac{\log(x)}{\log(b)}$$

One immediate consequence of the change of base formula is we can solve $b^x = A$ for x via

$$b^x = A \iff x = \frac{\log(A)}{\log(b)}$$

A second consequence is

$$\frac{d}{dx}[\log_b(x)] = \frac{d}{dx}\left[\frac{\log(x)}{\log(b)}\right] = \frac{1}{x \log(b)}$$