Notes on partial fractions

Before we use partial fractions as a method for integrating rational functions (ratio of two polynomial) there is the problem of division. For Pete's sake do it the cowboy way, not the donkey way. Use synthetic division

1.
$$\int \frac{3x^3 - 5x^2 + x - 2}{x - 2} dx$$

From the video we see

$$\frac{3x^3 - 5x^2 + x - 2}{x - 2} = 3x^2 + x + 3 + \frac{4}{x - 2}$$

solving

$$\int \frac{3x^3 - 5x^2 + x - 2}{x - 2} dx = x^3 + \frac{x^2}{2} + 3x + 4\ln(|x - 2|)$$

DO NOT put $u = x - 2$ and then divide. You will get an answer, but it will all be in terms of $x - 2$ rather than x

2.
$$\int \frac{x + 3}{x - 1} dx$$

You can divide if you like using synthetic division, but in this example that is the donkey way. You know the quotient will be 1, i.e $\frac{x+3}{x-1} = 1 + \frac{\text{remainder}}{x-2}$, only question is what is the remainder, so force the 1. Cowboy way:

$$\frac{x + 3}{x - 1} = \frac{x - 1 + 4}{x - 1} = 1 + \frac{4}{x - 1}$$

making

$$\int \frac{x + 3}{x - 1} dx = x + 4\ln(|x - 1|) + C$$

3.
$$\frac{x - 2}{x + 5} = \frac{x + 5 - 7}{x + 5} = 1 - \frac{7}{x + 5}$$

4.
$$\frac{2x + 9}{x + 3} = \frac{2x + 6 + 3}{x + 3} = 2 + \frac{3}{x + 3}$$

And the integrals of the above are now very easy.

Partial fractions

5. We start with the easiest type.

$$\frac{2}{x+1} - \frac{3}{x+2} = \frac{1-x}{(x+1)(x+2)}$$

by addition, but the left hand side is what we can easily integrate, whereas we cannot on the right. So the question is not "how do I add fractions?" which no one knows anyway, even in calc 2, but how do I "decompose"

$$\frac{1-x}{(x+1)(x+2)}$$

We know the denominators will be $x+1$ and $x+2$ so write

$$\frac{A}{x+1} + \frac{B}{x+2} = \frac{1-x}{(x+1)(x+2)}$$

and solve for A, B There is an easy way, a real easy way, and an annoying way. Unfortunately sometimes you need the annoying way, but we will start with the easy way. Add on the left, match the numerators

$$A(x+2) + B(x+1) = 1 - x$$

and this must be true **for all values of x**.

In particular it is true if $x = -1$ in which case we get

$$A(-1+2) + B(-1+1) = 1 - (-1) \iff A = 2$$

It should be obvious why we picked $x = -1$ and it also should then be clear then now we pick $x = -2$ and cut to the chase

$$B(-2+1) = 3 \iff B = -3$$

and our answer is the one we already knew.

6. Another easy one

$$\frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

so

$$A(x+2) + B(x-1) = 3$$

put $x = 1$ get $3A = 3 \iff A = 1$ put $x = -2$ get $-3B = 3 \iff B = -1$ making

$$\frac{3}{(x-1)(x+2)} = \frac{1}{x-1} - \frac{1}{x+2}$$

Real easy (cowboy way) is explained here

Start with

$$\frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

To find A, cover up the factor under the A, put $x = 1$ in

$$\frac{3}{\cancel{(x-1)}(x+2)}$$

get $A = \frac{3}{3} = 1$ then to find B do the same

$$B = \frac{3}{(-2-1)\cancel{(x+2)}} = -1$$

7.

$$\frac{3x^2 - x - 5}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1}$$

Notice that $x^2 + 1$ is irreducible (unless you want to mess with complex numbers) and that since it has degree two the numerator has degree one.

$$A(x^2+1) + (Bx + C)(x-2) = 3x^2 - x - 5$$

Now there are several ways to proceed. On annoying way it so multiply out, combine like terms, equate like coefficients and solve

$$Ax^2 + A + Bx^2 - 2Bx + Cx - 2C = 3x^2 - x - 5$$

$$(A+B)x^2 + (C - 2B)x + (A - 2C) = 3x^2 - x - 5$$

making

$$A + B = 3, C - 2B = -1, A - 2C = -5$$

and you have a nice 3 by 3 system to slolve....

Instead start with

$$A(x^2+1)+(Bx+C)(x-2)=3x^2-x-5$$

put $x=2$ get $5A=5 \iff A=1$

Now either by looking at the system above, or with your eyeballs see that $A+B=3$ making $B=2$

Find C in a similar fashion. This is a good way to avoid solving systems of equations. Also notice that once you have A,B you could write

$$x^2+1+(2x+C)(x-2)=3x^3-x-5$$

and then pick any value you like for x except of course 2. For example if $x=0$ you would get

$$1-2C=-5 \iff C=3$$

Now

$$\int \frac{3x^2-x-5}{(x-2)(x^2+1)}dx=\int \frac{3dx}{x-2}+\int \frac{2xdx}{x^2+1}+\int \frac{3dx}{x^2+1}=3\ln(|x-2|)+\ln(x^2+1)+3\tan^{-1}(x^2+1)+C$$

I finished this one to remind you that just because there is a denominator does not mean you get a log.

8. Repeated factors. Just like when you add

$$\frac{1}{2}+\frac{2}{3}+\frac{5}{9}$$

the denominator will be 2×3^2 when you decompose

$$\frac{4x^2-9x-4}{(x+1)(x-2)^2}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

Here

$$A(x-2)^2+B(x+1)(x-2)+C(x+1)=4x^2-9x-4$$

$x=2$ finds C, $x=-1$ gives A and to find B put in A,C and any value for x you choose.