Trigonometric Substitutions
Math 121 Calculus II
D Joyce, Spring 2013

Now that we have trig functions and their inverses, we can use trig subs. They're special kinds of substitution that involves these functions. For these, you start out with an integral that doesn't have any trig functions in them, but you introduce trig functions to evaluate the integrals. These depend on knowing

(1) the Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta$$

(2) the definitions

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

(3) the derivatives

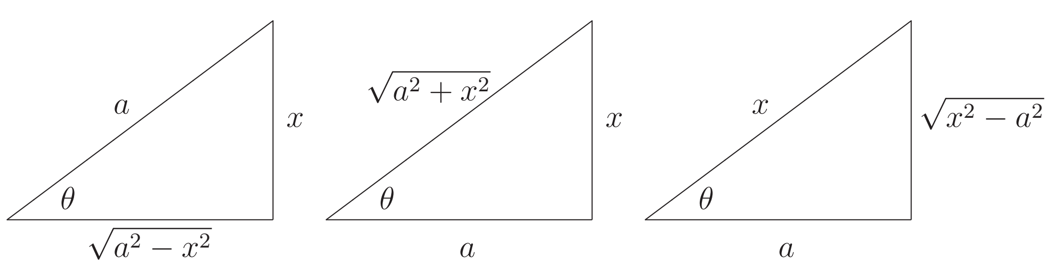
$$(\sin\theta)' = \cos\theta \qquad (\cos\theta)' = -\sin\theta$$
$$(\tan\theta)' = \sec^2\theta \qquad (\sec\theta)' = \sec\theta\tan\theta$$

There are three kinds of trig subs. You use them when you see as part of the integrand one of the expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, where a is some constant. In each kind you substitute for x a certain trig function of a new variable θ. The substitution will simplify the integrand since it will eliminate the square root. Here's a table summarizing the substitution to make in each of the three kinds.

If use see	use the sub	so that	and
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$dx = a\cos\theta\,d\theta$	$\sqrt{a^2 - x^2} = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta\,d\theta$	$\sqrt{a^2 + x^2} = a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta\,d\theta$	$\sqrt{x^2 - a^2} = a\tan\theta$

In each line, the last entry follows from the second entry by one of the Pythagorean identities.

There are also right triangles you can draw to make the connections between x, a, and θ. The three triangles below refer to the three trig subs, respectively.



Thank you D Joyce for saving me hours of work.

Recall (perhaps) from trig that if we have $\sqrt{16-x^2}$ and replace x by $4\sin(\theta)$ you have

$$\sqrt{16-(4\sin(\theta))^2} = \sqrt{16-16\sin^2(\theta)} = \sqrt{16(1-\sin^2(\theta))} = \sqrt{16\cos^2(\theta)} = 4\cos(\theta)$$

Well, actually you would get $4|\cos(\theta)|$ but no matter, restrict θ to make cosine positive. What does it buy us? We used to have a square root, now there is not one. Then again we used to have an x and now we have a function of θ. We will use this trick to get rid of square roots to compute integrals using "trig substitution". The above table explains exactly

what substitution you need in each circumstance, and also what you get as the derivative and what you get as the radical, so there is no need to do the algebra/trig each time. That is, if you see $\sqrt{16-x^2}$ you know immediately $x = 4\sin(\theta), dx = 4\cos(\theta)d\theta$ and the radical becomes $4\cos(\theta)$ There are lots of moving parts here; this is the most difficult integration because once we make the substitution, integrate, then we have to go back to x The bottom triangles will show us how.

1.
$$\int \frac{x^2dx}{\sqrt{16-x^2}}$$
We have $x = 4\sin(\theta), dx = 4\cos(\theta)d\theta, \sqrt{16-x^2} = 4\cos(\theta)$ we also know $\theta = \sin^{-1}\left(\frac{x}{4}\right)$ meaning we will use the triangle on the left with $a = 4$

$$\int \frac{x^2dx}{\sqrt{16-x^2}} = \int \frac{16\sin^2(\theta)4\cos(\theta)d\theta}{4\cos(\theta)} = 16\int \sin^2(\theta)d\theta = \frac{16}{2}\left(\theta - \sin(\theta)\cos(\theta)\right)$$

The last equal sign because

$$\int \sin^2(\theta) = \frac{\theta}{2} - \frac{\sin(2\theta)}{4}$$

as explained in your homework, and the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ OK great, now we have an answer, but only in terms of θ. But $\theta = \sin^{-1}\left(\frac{x}{4}\right)$ and from the triangle we see $\cos(\theta) = \frac{\sqrt{16-x^2}}{4}, \sin(\theta) = \frac{x}{4}$ Putting it all together we get

$$\int \frac{x^2dx}{\sqrt{16-x^2}} = 8\left(\sin^{-1}\left(\frac{x}{4}\right) - \frac{\sqrt{16-x^2}}{4} \times \frac{x}{4}\right) = 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16-x^2} + C$$

2.

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx$$

Middle row:

$$x = 2\tan(\theta), dx = 2\sec^2(\theta)d\theta, \sqrt{x^2+4} = 2\sec(\theta), \theta = \tan^{-1}\left(\frac{x}{2}\right)$$

Then

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta \cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta = -\frac{1}{4\sin\theta}$$

The last equal sign by the mental u-sub $u = \sin\theta$ Not that it makes any difference but we could write this as $-\dfrac{\csc(\theta)}{4}$ From the middle triangle we get $\csc(\theta) = \frac{\sqrt{4+x^2}}{x}$ Giving

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = -\frac{\sqrt{x^2+4}}{4x} + C$$