Recall the product rule

$$(fg)' = fg' + gf'$$

in whatever order you choose since multiplication and addition is commutative.

1.

$$(\sin(x) - x\cos(x))' = \sin'(x) - (x'\cos(x) + x\cos'(x)) = \cos(x) - (\cos(x) - x\sin(x)) = x\sin(x)$$

and therefore

$$\int x\sin(x)dx = \sin(x) - x\cos(x) + c$$

2.

$$\left(6x^6\ln(x) - x^6\right)' = 6(x^6)'\ln(x) + 6x^6\ln'(x) - (x^6)' = 36x^5\ln(x) + \frac{6x^6}{x} - 6x^5 = 36x^5\ln(x)$$

and therefore

$$\int x^5\ln(x) = \frac{1}{36}\left(6x^6\ln(x) - x^6\right) + C$$

3.

$$(x\ln(x) - x))' = x'\ln(x) + x\ln'(x) - x' = \ln(x) + \frac{x}{x} - 1 = \ln(x)$$

and therefore

$$\int \ln(x)dx = x\ln(x) - x + c$$

4.

$$(e^x(\sin(x) - \cos(x)))' = e^x(\cos(x) + \sin(x)) + e^x(\sin(x) - \cos(x)) = 2e^x\sin(x)$$

and so

$$\int e^x\sin(x)dx = \frac{1}{2}(e^x(\sin(x) - \cos(x)))$$

The question is, if you don't know the answer to the integral already, how to you arrive at it. Starting with the product rule

$$(fg)' = fg' + gf'$$

taking anti derivatives on both sides we get

$$fg = \int fg' + \int gf'$$

or

$$\int fg' = fg - \int gf'$$

called"integration by parts" usually written as

$$\int udv = uv - \int vdu$$

It is your job to pick which one is u and which is dv which requires (only a little) practice.

1.

$$\int x\sin(x)dx$$

put $u = x, dv = \sin(x)dx$. Why not the other way? Good question. Because $du = dx, v = -\cos(x)$ and you get a simpler integral. If you pick $dv = x$ then $v = \frac{x^2}{2}$ making things worse. So

$$\int \overbrace{x}^{u}\overbrace{\sin(x)dx}^{dv} = \overbrace{x}^{u}\overbrace{(-\cos(x)}^{v} - \int \overbrace{(-\cos(x)}^{v}\overbrace{dx}^{du}$$

$$= \sin(x) - x\cos(x) + C$$

2.

$$\int x^5\ln(x)dx$$

put $u = \ln(x), dv = x^5dx$ which might not have been the first guess, but now

$du = \frac{1}{x}dx, v = \frac{x^6}{6}$ giving

$$\int x^5\ln(x)dx = \frac{x^6}{6}\ln(x) - \int \frac{x^6}{6}\frac{1}{x}dx = \frac{1}{6}x^6\ln(x) - \frac{1}{6}\int x^5dx = \frac{1}{36}\left(6x^x\ln(x) - x^6\right) + C$$

3.

$$\int \ln(x)dx$$

Memorize this one, but the trick is to put $u = \ln(x), dv = dx, du = \frac{1}{x}dx, v = x$

$$\int \ln(x)dx = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - \int dx = x\ln(x) - x + C$$

4.

$$\int e^x \sin(x)dx$$

This type always seems confusing because there is a minuscule amount of elementary algebra at the end. It is not clear what to use for u or dv because nothing will become any simpler than what you have. Derivatives and anti-derivative of e^x are e^x so that wont change anything, and derivatives and anti - derivatives of $\sin(x)$ are $\cos(x), -\sin(x), -\cos(x), \sin(x)$ in some order. Therefore, in fact it does not matter what you choose for u or dv

I'll pick $u = \sin(x), dv = e^x dx$ making $du = \cos(x)dx, v = e^x$

$$\int e^x \sin(x)dx = e^x \sin(x) - \int e^x \cos(x)dx$$

And you can see we have accomplished nothing as the second integral is no better than the first. Integrate by parts again, with $u = \cos(x), dv = e^x, du = -\sin(x)dx, v = e^x$
We get

$$\int e^x \sin(x)dx = e^x \sin(x) - \int e^x \cos(x)dx = e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)dx$$

Check the sign of the last integral, it is correct. Let's rewrite without the middle part:

$$\underbrace{\overset{\text{original integral}}{\int e^x \sin(x)dx}} = e^x \sin(x) - e^x \cos(x) - \underbrace{\overset{\text{same integral!}}{\int e^x \sin(x)dx}}$$

Now comes the elementary algebra part. Add $\int e^x \sin(x)dx$ to both sides, divide by 2

$$2\int e^x \sin(x)dx = e^x(\sin(x) - \cos(x))$$

$$\int e^x \sin(x) = \frac{1}{2}\left(e^x\left(sin(x) - \cos(x)\right)\right) + C$$

This might look somewhat cumbersome because you have to integrate by parts twice, then some algebra. But imagine...

5.

$$\int x^4 \sin(x)dx$$

where you would have to integrate by parts 5 times to get rid of x^4

$$x^4 \to 4x^3 \to 12x^2 \to 24x \to 24$$

and also keep track of a boat load of plus an minus signs. Fortunately there is a very easy way to do this (and the one above) often called D-I or "tabular" method, well explained here