Week 11 Description
11.9
11.10
11.9: Representing a function as a power series. Most of this section uses the gimmick that we know

$$\sum x^n = \frac{1}{1-x} \text{ for } |x| < 1$$

1. Replacing x by $-x$ we get

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

2. Replacing x by $-x^2$ we get

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

3. And even

$$\begin{aligned} \frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n \end{aligned}$$

   For $|x| < 2$

4. Then how easy is

$$\frac{x^4}{2+x} = x^4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+4}$$

Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R

In plain (math) English this says we can differentiate and integrate term by term. Remember that polynomials are the easiest functions to differentiate and integrate, so it is easy here too.

5. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$

Taking derivatives we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

Notice that writing out the terms makes it clear what is going on, but for+ taking the derivative it is actually easier to go from $\sum_{n=0}^{\infty} x^n$ to $\sum_{n=1}^{\infty} nx^{n-1}$

6. We know from above

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

Integrating term by term we get

$$\log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Check that the first sum is what you get using the power rule, the second sum is the same as the first and that the constant which should be $+C$ is 0

11.10 May be the hardest section (other than trig subs). It concerns taking any function and trying to represent it as a power series. That is, given any f can we write

$$f(x) = f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

The book has a good explanation of why, if this is possible, the coefficients must be

$$c_n = \frac{f^{(n)}(a)}{n!}$$

That is, the nth coefficient is the nth derivative evaluated at a divided by $n!$

We can check the first few.

Since $f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + c_3(a-a)^3 + c_4(a-a)^4 + \cdots = c_0$ the constant is obvious. Taking derivatives we get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

Making $f'(a) = c_1$

The factorials come in because taking successive derivatives means for example that $f''(a) = 2c_2, f'''(a) = 3 \times 2c_3$ and so on.

7. Easiest example and the mother of all Maclaurin series (where $a = 0$) is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

   This is so easy because the derivative of e^x is e^x so each successive derivative is the same, and $e^0 = 1$ making each coefficient $\frac{1}{n!}$

8. Next easiest example is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

   This is easy because the successive derivatives of $\sin(x)$ are

$$\cos(x), -\sin(x), -\cos(x), \sin(x)$$

   and therefore at $a = 0$ you get a repeating pattern of

$$0, 1, 0, -1, 0, \cdots$$

   We can remember this one because it alternates and sine is an odd function, here all the terms have odd degree, which is not a coincidence.

9. Could repeat the process for

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

   or take the derivative of the series for sine term by term. Notice once again that cosine is even and all the powers are even.

The most common examples are on in table 1 in 11.10