Week 10 Description
11.6
11.8
11.6 On Absolute Convergence, ratio and root test for convergence.
1) Definition A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

1. For example
$$\sum \frac{(-1)^n}{n^2}$$
is absolutely convergent because
$$\sum \frac{1}{n^2}$$
converges, but

2. $$\sum \frac{(-1)^n}{n}$$
is not absolutely convergent because
$$\sum \frac{1}{n}$$
is the divergent harmonic series.

If the series converges but not absolutely, it is called "conditionally convergent" like the alternating harmonic series above.

The notes below in the this module are much better and easier to understand than the book. You should be able to do the problems in your head after reading it.

11.8 on Power series is a bit more difficult.

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Which is not as bad as it looks, these are just like polynomials, only the degree is not finite.

A power series could be "centered at a" and look like

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

The natural domain of any polynomial is all real numbers, but that is not the case with power series. For example we know the geometric series $\sum x^n$ converges to $\dfrac{1}{1-x}$ only if $|x| < 1$ otherwise it diverges. We will be concerned with what possible value of x can be used so that the power series converges.

Theorem: For a given power series $\sum_{n=0} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$

(ii) The series converges for all x

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

We will be concerned with finding R the "radius of convergence"

3. The radius of convergence for $\sum x^n$ is 1, and the interval of convergence is $(-1, 1)$

4. Easy famous example: Find the radius of convergence of

$$\sum \frac{x^n}{n!}$$

Before we begin a quick side trip to recall that $n!$ is read "n factorial" and

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1$$

For example,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Check for yourself that

$$\frac{6!}{5!} = 6 \text{ and } \frac{(n+1)!}{(n-1)!} = n(n+1)$$

Using the ratio (or root test) we compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| = \frac{1}{n+1}\,|x|$$

and therefore

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$$

Notice that the limit is in n and does not depend on x. Since $0 < 1$ we know this will converge no matter what x is, so radius of convergence is ∞ and interval of convergence is $\mathbb{R}$

5. Harder example: Find the radius of convergences of

$$\sum \frac{(x-2)^n}{n3^n}$$

We use the ratio test to get

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{(n+1)3^{n+1}} \times \frac{n3^n}{(x-2)^n}\right| = \frac{1}{3} \times \frac{n}{n+1}\,|x-2|$$

Check this algebra. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\,|x-2|$$

Set

$$\frac{1}{3}\,|x-2| < 1 \iff |x-2| < 3$$

Gives $R = 3$ as the radius of convergence.

For the interval of convergence, solve

$$|x-2| < 3 \iff -1 < x < 5$$

Put $x = -1$ get

$$\sum \frac{(-1-2)^n}{n3^n} = \sum \frac{(-3)^n}{n3^n} = \sum \frac{(-1)^n}{n}$$

which is the convergent alternation harmonic series. Put $x = 5$ get the divergent harmonic series (try it) making the interval of convergence $[-1, 5)$