Two simple example of a u-substitution:

1.
$$\int \sqrt{\sin(x)}\cos(x)dx$$
The substitution is $u = \sin(x), du = \cos(x)dx$ giving
$$\int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{3}\sqrt{\sin(x)^3} + C$$

2.
$$\int \sin(\pi x)dx$$
the mental substitution is $u = \pi x, du = \pi dx \iff \frac{du}{\pi} = dx$ giving
$$\frac{1}{\pi}\int \sin(u)du = -\frac{1}{\pi}cos(u) = -\frac{1}{\pi}cos(\pi x) + C$$

Examples of substitution $u = g(x)$ where $g'(x)$ does not appear in the integrand. The idea is to make $u = g(x)$ and so long as g is one to one we can solve for x
$$u = g(x) \iff x = g^{-1}(u)$$
And
$$dx = \left(g^{-1}\right)'(u)du$$
Apply to the integral $\int f(g(x))dx$ gives
$$\int f(g(x))dx = \int f(u)\left(g^{-1}\right)'(u)du$$
On the other hand, if we apply the straightforward substitution
$$\begin{aligned} u &= g(x) \\ du &= g'(x)dx \end{aligned}$$
to the same integral,
$$\int f(g(x))dx = \int f(g(x)) \cdot \frac{1}{g'(x)} \cdot g'(x)dx$$
we obtain
$$\int f(u) \cdot \frac{1}{g'\left(g^{-1}(u)\right)}du$$
The integrals (1) and (2) are identical, since $\left(g^{-1}\right)'(u) = 1/g'\left(g^{-1}(u)\right)$ (why?)

1.

$$\int e^{\sqrt{x}}dx$$

Here we take

$$u = \sqrt{x} \iff x = u^2$$
$$dx = 2udu$$

giving

$$2\int ue^u du$$

This requires integration by parts, stay tuned.

2.

$$\int_0^9 \sqrt{4-\sqrt{x}}dx$$

The substitution is $u = 4 - \sqrt{x}$ and now change the limit of integration so we don't have to substitute back $u(0) = 4 - \sqrt{0} = 4, u(9) = 4 - \sqrt{9} = 1$ Solve $u = 4 - \sqrt{x}$ for x

$u = 4 - \sqrt{x}$
$u - 4 = -\sqrt{x}$
$(u-4)^2 = x$
$2(u-4)du = dx$

$$\int_0^9 \sqrt{4-\sqrt{x}}dx = 2\int_4^1 \sqrt{u}(u-4)du = 2\int_4^1 u^{\frac{3}{2}} - 4u^{\frac{1}{2}}du$$

etc.

3.

$$\int \frac{x^2+4}{x+2}dx$$

here the substitution $u = x + 2$ gets rid of the pesky denominator

$u = x + 2$
$u - 2 = x$
$du = dx$
$(u-2)^2 + 4 = x^2 + 4$

$$\int \frac{(u-2)^2+4}{u}du = \int \left(u - 4 + \frac{8}{u}\right)du = \frac{u^2}{2} - 4u + 8\ln(u)$$

etc

HOWEVER although this is probably the wolfram solution it is a lot easier to divide, especially since synthetic division is very easy. It saved the algebra steps here: $\frac{(u-2)^2+4}{u} = u - 4 + \frac{8}{u}$ AND you do not have to go back and replace u by $x+2$

$$\frac{x^2+4}{x+2} = x - 2 + \frac{8}{x-2}$$

and now the integral is very easy.

4.

$$\int \frac{5}{1+2x^2} dx$$

The important part here is to recall that the derivative of arctangent is $\frac{1}{1+x^2}$ which is not exactly what we have, so the substitution is $u = \sqrt{2}x$

$u = \sqrt{2}x$
$\frac{u}{\sqrt{2}} = x$
$\frac{du}{\sqrt{2}} = dx$

$$\frac{5}{\sqrt{2}} \int \frac{du}{1+u^2} = \frac{5}{\sqrt{2}} \tan^{-1}(u) = \frac{5}{\sqrt{2}} \tan^{-1}(\sqrt{2}x)$$