The Root Test

Lecture Notes

So far, we have learned how to use the limit comparison test to determine whether a series converges or diverges. The idea of the limit comparison test is that a series will converge as long as its terms go to zero quickly enough.

Unfortunately, there are many series for which it is difficult to tell how quickly the terms go to zero. For example, consider the series

$$\sum_{n=1}^{\infty} \frac{n^3}{2^n}.$$

Since $2^n \gg n^3$, we know that the terms of this series approach zero. However, it is not clear whether they go to zero quickly enough for the series to converge.

It turns out that this series *does* converge. In fact, the n^3 doesn't make much difference—this series is only slightly larger than $1/2^n$:

$$\frac{1}{2^n} \ll \frac{n^3}{2^n} \ll \cdots \ll \frac{1}{(1.999)^n} \ll \frac{1}{(1.99)^n} \ll \frac{1}{(1.5)^n} \ll \cdots$$

The exponential 2^n is so overwhelmingly large that n^3 in the numerator hardly makes a difference. From the point of view of exponential functions, an n^3 isn't very different from a constant.

In general, we say that a series ***converges exponentially*** if it lies within the exponential portion of the hierarchy. You can determine whether a series converges exponentially by focusing on the exponential factors, treating smaller factors as though they were constant.

EXAMPLE 1 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{6^n\sqrt{n}}$ converges or diverges.

SOLUTION Though the $\sqrt{n}$ goes to infinity, it does not grow quickly enough to make much of a difference. Only the 2^n and 6^n will affect the convergence. Since the series

$$\sum_{n=1}^{\infty} \frac{2^n}{6^n} = \sum_{n=1}^{\infty} \frac{1}{3^n}$$

converges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{6^n\sqrt{n}}$ must converge as well. ■

When you focus on the exponential terms of a series, the result is usually a geometric series. Make sure to remember the rule for convergence of a geometric series:

CONVERGENCE TEST FOR GEOMETRIC SERIES

The geometric series

$$\sum_{n=0}^{\infty} ar^n$$

converges if $|r| < 1$, and diverges if $|r| \geq 1$.

EXAMPLE 2 Determine whether the following series converge or diverge:

$$\text{(a)} \quad \sum_{n=0}^{\infty} \frac{2^n n^2}{3^n} \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{5^n}{3^n (n^4 + 2)} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{e^n}{3^n\, n^2 \ln n}$$

SOLUTION In each case, the key is to focus on the exponential factors. Because the other factors grow so slowly, they will not play any role in the convergence.

(a) Ignoring the n^2, this is the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$. Since this geometric series has a common ratio of $r = 1$, it converges.

(b) Ignoring the $n^4 + 2$, this is the series $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{5}{3}\right)^n$. Since the common ratio $r = 5/3$ of this geometric series is greater than 1, the series diverges. (Indeed, the terms of this series do not even approach zero.)

(c) Ignoring the n^2 and the $\ln n$, this is the series $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{e}{3}\right)^n$, which has a common ratio of $r = e/3$. Since $e/3 < 1$, this series converges. ■

The Root Test

The ***root test*** is a more sophisticated way to determine whether a series converges exponentially:

THE ROOT TEST

Let $\sum a_n$ be a series with positive terms, and let

$$r = \lim_{n \to \infty} \sqrt[n]{a_n}.$$

(a) If $r < 1$, then the series $\sum a_n$ converges.

(b) If $r > 1$, then the series $\sum a_n$ diverges.

(c) If $r = 1$, then the root test is inconclusive.

The idea of taking the nth root is that it picks out the base of an exponential. For example,

$$\sqrt[n]{2^n} = 2 \qquad \text{and} \qquad \sqrt[n]{\frac{1}{3^n}} = \frac{1}{3}.$$

For a geometric series, the value of r will be the common ratio of the series. For other series, it represents the common ratio of the "closest" geometric series on the hierarchy.

EXAMPLE 3 Use the root test to determine whether the following geometric series converge:

$$\text{(a)} \quad \sum_{n=0}^{\infty} \frac{3^n}{5^n} \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{2^{2n}3^n}{10^n} \qquad \text{(c)} \quad \sum_{n=0}^{\infty} \frac{4^n}{3^{n+2}}$$

SOLUTION

(a) We have

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{3^n}}{\sqrt[n]{5^n}} = \frac{3}{5}.$$

Since $3/5 < 1$, series (a) converges;

(b) Since taking the nth root divides any exponent by n, the nth root of 2^{2n} is simply 2^2. Therefore,

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{2^{2n}3^n}{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{2^{2n}}\,\sqrt[n]{3^n}}{\sqrt[n]{10^n}} = \frac{(2^2)(3)}{(10)} = \frac{12}{10}.$$

Since $12/10 > 1$, this series diverges.

(c) The key fact here is that

$$\lim_{n\to\infty} \sqrt[n]{3^{n+2}} = 3$$

This makes sense, since 3^{n+2} is an exponential with a base of 3. Algebraically, this can be seen as follows:

$$\lim_{n\to\infty} \sqrt[n]{3^{n+2}} = \lim_{n\to\infty} \left(3^{n+2}\right)^{1/n} = \lim_{n\to\infty} 3^{(n+2)/n} = 3^1 = 3.$$

Therefore,

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{4^n}{3^{n+2}}} = \lim_{n\to\infty} \frac{\sqrt[n]{4^n}}{\sqrt[n]{3^{n+2}}} = \frac{4}{3}.$$

Since $4/3 > 1$, this series diverges. ■

The root test can be used for many series that are not geometric. For such series, it necessary to evaluate limits of nth roots of more complicated expressions. The following rules are often helpful:

RULES FOR NON-EXPONENTIALS

1. $\lim_{n\to\infty} \sqrt[n]{C} = 1$, for any positive constant C.
2. $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$, for any positive exponent p.
3. $\lim_{n\to\infty} \sqrt[n]{\ln n} = 1$.
4. $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$.
5. $\lim_{n\to\infty} \sqrt[n]{n} = \infty$.

EXAMPLE 4 Use the root test to determine whether the following series converge:

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n n^3}{3^n} \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{5^n}{3^n(n^4+2)} \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{5^n}{2^n n!}$$

SOLUTION

(a) We have

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{2^n n^3}{3^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{2^n}\,\sqrt[n]{n^3}}{\sqrt[n]{3^n}} = \frac{(2)(1)}{(3)} = \frac{2}{3}.$$

Since $2/3 < 1$, this series converges by the root test.

(b) We have

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{5^n}{3^n(n^4+2)}} = \lim_{n\to\infty} \frac{\sqrt[n]{5^n}}{\sqrt[n]{3^n}\,\sqrt[n]{n^4+2}} = \frac{(5)}{(3)(1)} = \frac{5}{3}.$$

Since $5/3 > 1$, this series diverges by the root test.

(d) We have

$$r = \lim_{n\to\infty} \sqrt[n]{\frac{5^n}{2^n n!}} = \lim_{n\to\infty} \frac{\sqrt[n]{5^n}}{\sqrt[n]{2^n}\,\sqrt[n]{n!}} = \frac{(5)}{(2)(\infty)} = 0.$$

Since $0 < 1$, this series converges by the root test. Note that the $n!$ was the only factor of this series that made any difference. Because $n!$ is so much larger than any exponential, it completely overwhelms the other factors. ■

At this point, you may be puzzled by some of our assertions regarding limits of nth roots. For example, we have stated that

$$\lim_{n\to\infty} \sqrt[n]{n^p} \;=\; 1$$

for any positive exponent p. Why would this be the case?

The way to understand these limits is to think about the asymptotic hierarchy:

$$\ln n \;\ll\; \cdots \underbrace{\;\ll\; \sqrt{n} \;\ll\; n \;\ll\; n^2 \;\ll\;}_{\text{polynomials}} \cdots \underbrace{\;\ll\; 2^n \;\ll\; 3^n \;\ll\; 4^n \;\ll\;}_{\text{exponentials}} \cdots \;\ll\; n! \;\ll\; n^n$$

The nth root of an exponential is just the base of the exponential:

$$\begin{array}{ccccccccccccc}
\cdots & \ll & (1.01)^n & \ll & (1.1)^n & \ll & 2^n & \ll & 3^n & \ll & 4^n & \ll & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & & 1.01 & & 1.1 & & 2 & & 3 & & 4 & & \cdots
\end{array}$$

Therefore, if a function is smaller than every exponential (like a power of n), then the limit of its nth root must be 1:

$$\begin{array}{ccccccccccccccccccc}
\cdots & \ll & n & \ll & n^2 & \ll & n^3 & \ll & \cdots & \ll & (1.01)^n & \ll & (1.1)^n & \ll & 2^n & \ll & 3^n & \ll & 4^n & \ll & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & & 1 & & 1 & & 1 & & \cdots & & 1.01 & & 1.1 & & 2 & & 3 & & 4 & & \cdots
\end{array}$$

Using the same reasoning, if a function is bigger than every exponential, then the limit of its nth root must be infinite:

$$\begin{array}{ccccccccccccccc}
\cdots & \ll & 2^n & \ll & 3^n & \ll & 4^n & \ll & 5^n & \ll & \cdots & \ll & n! & \ll & n^n \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\cdots & & 2 & & 3 & & 4 & & 5 & & & & \infty & & \infty
\end{array}$$

In summary, taking the limit of the nth root has the following effect:

$$\boxed{\begin{array}{ccccccccc}
\underbrace{\ln n \;\ll\; \cdots \;\ll\; \sqrt{n} \;\ll\; n \;\ll\; n^2 \;\ll\;} & \cdots \;\ll & 2^n & \ll & 3^n & \ll & 4^n & \ll\; \cdots & \underbrace{\;\ll\; n! \;\ll\; n^n} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & & 2 & & 3 & & 4 & \cdots & \infty
\end{array}}$$

Knowing this information makes it very easy to apply the root test to most series. Indeed, taking the nth root can usually be performed in one step. For example, the series

$$\sum_{n=1}^{\infty} \frac{5^n n^2 \ln n}{2^n 3^{n+1}}$$

converges, since

$$r \;=\; \frac{(5)(1)(1)}{(2)(3)} \;=\; \frac{5}{6} \;<\; 1.$$

For some series, though, the limit plays a more important role:

EXAMPLE 5 Use the root test to determine whether the following series converge:

$$\text{(a)}\quad \sum_{n=1}^{\infty}\left(\frac{1}{5}+\frac{1}{n}\right)^{n} \qquad \text{(b)}\quad \sum_{n=1}^{\infty}\left(\tan^{-1} n\right)^{n} \qquad \text{(c)}\quad \sum_{n=0}^{\infty}\frac{n^{n}}{2^{(n^2)}}$$

SOLUTION

(a) We have

$$r \;=\; \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{5}+\frac{1}{n}\right)^{n}} \;=\; \lim_{n\to\infty}\frac{1}{5}+\frac{1}{n} \;=\; \frac{1}{5}.$$

Since $1/5 < 1$, this series converges.

(b) We have

$$r \;=\; \lim_{n\to\infty}\sqrt[n]{\left(\tan^{-1} n\right)^{n}} \;=\; \lim_{n\to\infty}\tan^{-1} n \;=\; \frac{\pi}{2}.$$

Since $\pi/2 > 1$, this series diverges.

(c) We have

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^{n}}{2^{(n^2)}}} \;=\; \lim_{n\to\infty}\frac{\sqrt[n]{n^{n}}}{\sqrt[n]{2^{(n^2)}}} \;=\; \lim_{n\to\infty}\frac{n}{2^{n}} \;=\; 0.$$

Since $0 < 1$, this series converges. ■

By the way, it's important to remember that the root test is inconclusive when $r = 1$. Indeed, all of the following series have $r = 1$:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad \sum_{n=1}^{\infty}\frac{1}{n} \qquad \sum_{n=1}^{\infty}1 \qquad \sum_{n=1}^{\infty}n^3$$

The first series converges, while the remaining three all diverge. The only thing these series have in common is that none of them are exponential, and are therefore not susceptible to analysis by the root test.

EXERCISES

1–22 ■ Use the root test to determine whether the given series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^5}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^2}{n\,3^n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5\,2^n}{n^2+4n+1}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n\sqrt{n}}{2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n \ln n}{3^n\,n^3}$

7. $\displaystyle\sum_{n=1}^{\infty} (0.2)^n n^3$

8. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)\,3^{2n}}{n\,5^n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n/2}}{n^3\,2^n}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{5e^n}{2^{3n+1}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(2\pi)^n}{2^{2n-5}}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n-1}}{(n+1)^2\,4^{n+2}}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n+5}}{2^n 5^n n^3}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{5^{2n} e^n}{n!\sqrt{n}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n^3 e^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n+1}}{(\ln n)^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(\tan^{-1} n)^n \ln n}{\sqrt[3]{n}}$

20. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n-1}\right)^n$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\sqrt{n}}}{2^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n^2}}{n^n}$

23. For which of the following series is the root test inconclusive (that is, it fails to give a definite answer)?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}}{\sqrt{n}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$