Convergence and Divergence

Lecture Notes

It is not always possible to determine the sum of a series exactly. For one thing, it is common for the sum to be a relatively arbitrary irrational number:

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \;=\; 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots \;=\; 1.291286\ldots$$

The sum of this series isn't something simple like $\sqrt{2}$ or $\pi^2/6$ — it's just some arbitrary real number, whose digits can be determined only by adding together the terms of the series. (For example, the decimal approximation above was obtained by adding together the first hundred terms.)

We have encountered this sort of problem before. Recall that there is no way to find the exact value of the integral:

$$\int_0^1 e^{x^2}\, dx \;=\; 1.462652\ldots$$

The best you can do is a decimal approximation using rectangles or trapezoids.

For series, these sorts of problems are ubiquitous. There are very few general techniques for finding the exact sum of a series, and even relatively simple series such as $\sum_{n=1}^{\infty} 1/n^3$ cannot be summed exactly. One of the hardest problems in mathematics, the Riemann hypothesis, essentially just asks for what values of p the series

$$\eta(p) \;=\; \frac{1}{1^p} \;-\; \frac{1}{2^p} \;+\; \frac{1}{3^p} \;-\; \frac{1}{4^p} \;+\; \cdots$$

sums to exactly zero. This problem was first proposed nearly 150 years ago, and it *still has not been solved*. (In the year 2000, the Clay Mathematics Institute announced a $1 million prize for a solution to this problem.)

Because finding the exact sum of a series is so hard, we will usually *not* concern ourselves with adding up a given series exactly. Instead, we will focus on a much easier problem: can we at least figure out whether a given series converges?

Positive Series

For various reasons, it is simpler to understand convergence and divergence for series whose terms are all positive numbers. We shall refer to such series as **positive series**. Because each partial sum of a positive series is greater than the last, every positive series either converges or diverges to infinity. (As we shall see later on, series with negative terms have other possible behaviors.)

RULE FOR POSITIVE SERIES

If $\sum a_n$ is a positive series, then either

1. $\sum a_n$ converges to a positive number, or
2. $\sum a_n$ diverges to infinity.

We have seen many examples of convergent series, the most basic being:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

This series is geometric, with each term a constant multiple of the last. (In this case, each term is half as big as the previous one.) This repeated multiplication causes the terms of a geometric series to become small very quickly. For example, the 100th term of the above series is:

$$\frac{1}{2^{100}} = \frac{1}{1{,}267{,}650{,}600{,}228{,}229{,}401{,}496{,}703{,}205{,}376}$$

$$\approx 0.000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,789$$

We also know that a series diverges if its terms don't approach zero. For example:

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots = \infty$$

The terms in the above series get closer and closer to 1, causing the series to diverge to ∞.

However, there are also lots of divergent series whose terms do approach zero. Here is an illustrative example:

EXAMPLE 1 Consider the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots$$

Even though the individual terms of this series converge to zero, the sum of the entire series is infinite. To see this, consider what happens if we group similar terms together:

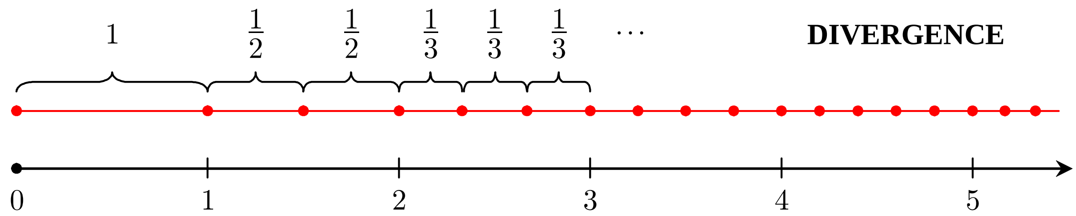
$$1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + \cdots$$

Because each group of terms adds up to 1, the total sum must be infinite:

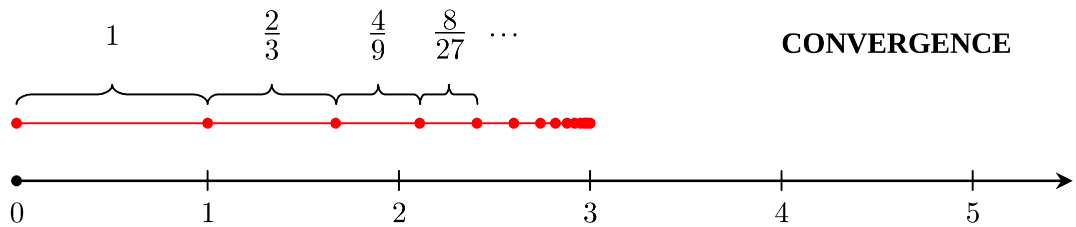
$$1 + 1 + 1 + 1 + \cdots = \infty. \qquad \blacksquare$$

The idea is that a series only converges if its *terms are small* (or become small *quickly*). For the series in the last example, the hundredth term is $1/14$, and the thousandth term is $1/45$ — much larger than the hundredth or thousandth term of a geometric series.

Here is a picture illustrating the sum of the series in the last example:



As you can see, the individual terms of the series are getting smaller, but not *quickly* enough for the sum to be finite. By contrast, here is a picture of the geometric series $1 + \frac{2}{3} + \frac{4}{9} + \cdots$:



The terms of this series become very small very quickly, forcing the sum to converge.

If you want to know whether a series $\sum a_n$ with positive terms converges, the main question is **how quickly** the terms a_n approach zero as $n \to \infty$.

The Comparison Test

The basic technique for understanding positive series is to **compare them with each other**. This is based on the following principle:

THE COMPARISON THEOREM

Let $\sum a_n$ and $\sum b_n$ be positive series, and suppose that

$$a_n \leq b_n$$

for each term. Then

$$\sum a_n \leq \sum b_n.$$

That is, if the terms of $\sum a_n$ are smaller than the corresponding terms of $\sum b_n$, then the sum of $\sum a_n$ must be less than the sum of $\sum b_n$.

EXAMPLE 2 Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges or diverges.

SOLUTION Recall that n^n becomes large very quickly as $n \to \infty$. Then the reciprocals $1/n^n$ must become small very quickly, which ought to cause the series to converge.

To show this, let's examine the first few terms of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \frac{1}{4} + \frac{1}{27} + \frac{1}{256} + \frac{1}{3125} + \cdots$$

As expected, the terms become very small, very quickly. For example, each term of this series is *smaller* than the corresponding term of the series $\sum 1/2^n$:

$$\begin{array}{ccccccccccc} 1 & + & \frac{1}{4} & + & \frac{1}{27} & + & \frac{1}{256} & + & \frac{1}{3125} & + & \cdots \\ 1 & + & \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + & \cdots \end{array}$$

It follows that **the sum of the top series must be smaller than the sum of the bottom series**:

$$\left(1 + \frac{1}{4} + \frac{1}{27} + \frac{1}{256} + \cdots\right) < \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = 2$$

This proves that the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges to some number smaller than two. ■

In general, if you know that a series converges, then any **smaller** series must converge as well. On the other hand, if you know that a series diverges, then any **larger** series must diverge as well. This is the basic test for convergence:

COMPARISON TEST

Let $\sum a_n$ and $\sum b_n$ be positive series.

1. If $\sum b_n$ converges and $a_n \leq b_n$, then $\sum a_n$ must converge as well.
2. If $\sum b_n$ diverges and $a_n \geq b_n$, then $\sum a_n$ must diverge as well.

EXAMPLE 3 Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$ converges or diverges.

SOLUTION We know that the sum of $1/2^n$ converges:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

How do things change when we introduce the extra factor of n?

The answer is that the new factor makes the denominator bigger:

$$n \cdot 2^n \;\geq\; 2^n$$

and therefore makes the whole fraction *smaller*:

$$\frac{1}{n \cdot 2^n} \;\leq\; \frac{1}{2^n}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, the smaller series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$ must converge as well. ■

The argument in the above example was entirely algebraic, and therefore somewhat abstract. In case you weren't convinced, here's a numerical comparison of the two series:

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} \;=\; \frac{1}{2} \;+\; \frac{1}{8} \;+\; \frac{1}{24} \;+\; \frac{1}{64} \;+\; \frac{1}{160} \;+\; \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \;=\; \frac{1}{2} \;+\; \frac{1}{4} \;+\; \frac{1}{8} \;+\; \frac{1}{16} \;+\; \frac{1}{32} \;+\; \cdots$$

Each term in the first series is smaller than the corresponding term of the second series. Since the second series adds up to 1, the first series must add up to some number less than 1.

In the next example, we use the comparison test to show that a series diverges:

EXAMPLE 4 The series

$$\sum_{n=2}^{\infty} \frac{1}{n} \;=\; \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

is known as the **harmonic series**. It is a very important fact that **the harmonic series diverges**. There are several different ways to show this, and one of them is to use a comparison.

What series should we compare to? Well, if we want to show that the sum of the harmonic series is infinite, we must show that its terms are *bigger* than the terms of another series that sums to infinity. This requires quite a bit of cleverness.

Here's one comparison that works:

harmonic series	$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$
new series	$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots$

As you can see, each term of the harmonic series is *bigger* than the corresponding term in the new series. But the new series diverges, which can be seen by grouping the terms together:

$$\frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.$$

Since the harmonic series is even larger than this divergent series, it must diverge as well. ■

Another way of thinking about the reasoning above is that, if we group together certain terms of the harmonic series, then the sum of each grouping is at least $1/2$:

$$\frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

$$\frac{1}{2} \qquad > \frac{1}{2} \qquad > \frac{1}{2} \qquad > \frac{1}{2}$$

In some sense, the harmonic series is the most basic series that diverges. Now that we know about it, we can use it for comparisons:

EXAMPLE 5 Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ converges or diverges.

SOLUTION Recall that $\ln n$ goes to infinity *very slowly* as $n \to \infty$. This makes the terms of the above series go to zero very slowly (e.g. the 1000th term of this series is slightly greater than $1/7$), which indicates that the series ought to diverge.

We can verify this by comparing with the harmonic series. It is easy to check that

$$\ln n \;\leq\; n$$

for any value of n, and so

$$\frac{1}{\ln n} \;\geq\; \frac{1}{n}.$$

Since the harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, the larger series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ must diverge as well. ■

P-Series

A ***p*-series** is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

where p is a constant. For example, when $p = 1$, this is the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} \;=\; 1 \;+\; \frac{1}{2} \;+\; \frac{1}{3} \;+\; \frac{1}{4} \;+\; \frac{1}{5} \;+\; \cdots \qquad (p=1)$$

When $p = 2$, it is the sum of the reciprocals of the squares:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; 1 \;+\; \frac{1}{2^2} \;+\; \frac{1}{3^2} \;+\; \frac{1}{4^2} \;+\; \frac{1}{5^2} \;+\; \cdots \qquad (p=2)$$

Higher values of p would correspond to cubes, fourth powers, and so forth. The exponent p could also be a fraction. For example, $p = 1/2$ gives the reciprocals of the square roots:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \;=\; 1 \;+\; \frac{1}{\sqrt{2}} \;+\; \frac{1}{\sqrt{3}} \;+\; \frac{1}{\sqrt{4}} \;+\; \frac{1}{\sqrt{5}} \;+\; \cdots \qquad (p=1/2)$$

The larger the value of p, the more quickly the terms of a p-series go to zero. The following theorem tells us how the convergence of a p-series depends on p:

CONVERGENCE OF P-SERIES

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for $p > 1$, and diverges for $p \leq 1$.

That is, the p-series $\sum_{n=1}^{\infty} 1/n^p$ behaves just like the improper integral

$$\int_1^{\infty} \frac{1}{x^p}\, dx,$$

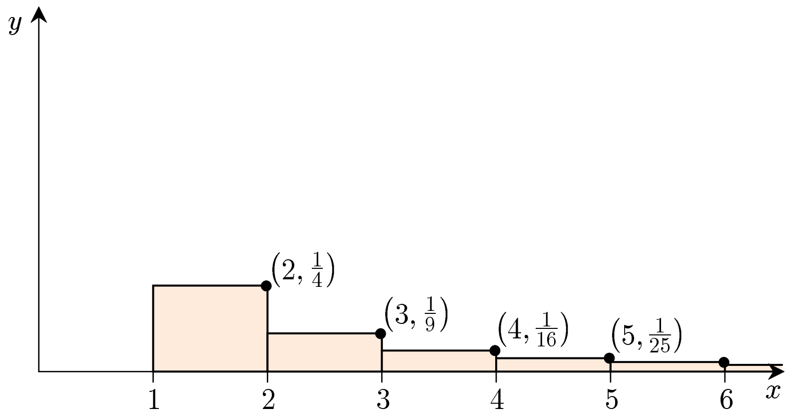
which is finite for $p > 1$, and infinite for $p \leq 1$. This similarity between p-series and improper integrals is not a coincidence. Indeed, the simplest way to show that a p-series converges is to **compare it with an integral**.

EXAMPLE 6 Consider the following series:

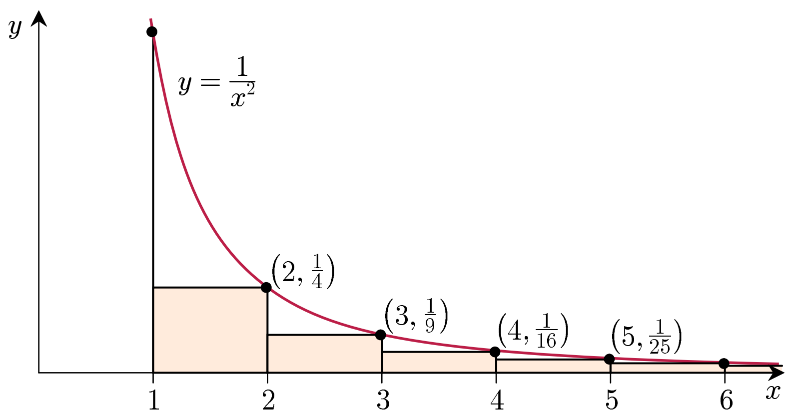
$$\sum_{n=2}^{\infty} \frac{1}{n^2} \;=\; \frac{1}{4} \;+\; \frac{1}{9} \;+\; \frac{1}{16} \;+\; \frac{1}{25} \;+\; \cdots$$

We shall show that this series converges by comparing it with an improper integral.

To start with, consider the following arrangement of rectangles:



The areas of the rectangles are $1/4$, $1/9$, $1/16$, and so on, with the total area being the sum of the series. However, all of the rectangles lie *under* the graph of $1/x^2$:



In particular, the total area of the rectangles must be *less* than the area under the curve. But the area under $1/x^2$ is finite:

$$\int_1^\infty \frac{1}{x^2}\,dx \;=\; 1$$

It follows that

$$\sum_{n=2}^\infty \frac{1}{n^2} \;<\; 1.$$

In particular, this series must converge. ■

Using this exact same argument, it is possible to show that

$$\sum_{n=2}^\infty \frac{1}{n^p} \quad<\quad \int_1^\infty \frac{1}{x^p}\,dx$$

for any positive exponent p. In particular, the p-series must converge whenever the integral does.

The Hierarchy of Series

Most of the series we have been discussing fit into a hierarchy:

$$\frac{1}{n^n} \ll \frac{1}{n!} \ll \cdots \ll \overbrace{\frac{1}{3^n} \ll \frac{1}{2^n}}^{\text{geometric series}} \ll \cdots \ll \overbrace{\frac{1}{n^2} \ll \frac{1}{n} \ll \frac{1}{\sqrt{n}}}^{p\text{-series}} \ll \cdots \ll \frac{1}{\ln n}$$

Each series in this hierarchy is the reciprocal of a function from the asymptotic hierarchy (see the *Limits at Infinity* notes). Because we have taken reciprocals, the order of the functions has reversed (so $1/\ln n$ is the largest, and $1/n^n$ is the smallest).

The series on the left side of this hierarchy converge (since they are the smallest), while the series on the right side diverge. The barrier between convergence and divergence is in the middle of the p-series:

$$\underbrace{\frac{1}{n^n} \ll \frac{1}{n!} \ll \cdots \ll \frac{1}{3^n} \ll \frac{1}{2^n} \ll \cdots \ll \frac{1}{n^2} \ll \frac{1}{n^{1.1}} \ll \cdots}_{\textbf{convergent}} \;\Big|\; \underbrace{\ll \frac{1}{n} \ll \frac{1}{\sqrt{n}} \ll \cdots \ll \frac{1}{\ln n}}_{\textbf{divergent}}$$

Note that the harmonic series is the first p-series that diverges.

Many complicated series can be handled by determining where they fit on the hierarchy:

LIMIT COMPARISON TEST

Let $\sum a_n$ and $\sum b_n$ be series of positive numbers, and suppose that:

$$a_n \sim b_n$$

Then the two series either both converge or both diverge.

Remember that $a_n \sim b_n$ means $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

EXAMPLE 7 Determine whether the following series converge or diverge:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{n^2+5} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{\sqrt{n^4-1}}{n^3} \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{n^5+2^n}{3^n}$$

SOLUTION For series (a), observe that:

$$\frac{1}{n^2+5} \sim \frac{1}{n^2}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, series (a) must converge as well.

For series (b), observe that:

$$\frac{\sqrt{n^4-1}}{n^3} \quad\sim\quad \frac{n^2}{n^3} \quad=\quad \frac{1}{n}$$

Since the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, series (b) must diverge as well.

For series (c), observe that:

$$\frac{n^5+2^n}{3^n} \quad\sim\quad \frac{2^n}{3^n} \quad=\quad \left(\frac{2}{3}\right)^n$$

Since $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ is a convergent geometric series, series (c) must converge as well. ■

Some series fall between the spots shown on the hierarchy. In this case, one must deduce where the series lies:

EXAMPLE 8 Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ converges or diverges.

SOLUTION Remember that $\ln n$ goes to infinity, but more slowly than *any* positive power of n. For example,

$$\ln n \quad\ll\quad n^{0.01}.$$

Therefore, the $\ln n$ in the numerator has no more effect than a $n^{0.01}$ would:

$$\frac{\ln n}{n^2} \quad\ll\quad \frac{n^{0.01}}{n^2} = \frac{1}{n^{1.99}}.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1.99}}$ converges, the smaller series $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ must converge as well. ■

Based on our reasoning in the last example, $\dfrac{\ln n}{n^2}$ must fit into the hierarchy *after* $\dfrac{1}{n^2}$, but *before* any smaller power of n:

$$\cdots \ll \frac{1}{n^2} \ll \boxed{\frac{\ln n}{n^2}} \ll \cdots \ll \frac{1}{n^{1.999}} \ll \frac{1}{n^{1.99}} \ll \frac{1}{n^{1.9}} \ll \frac{1}{n} \ll \cdots$$

There are many other spots like this in the hierarchy. For example, $\dfrac{n}{3^n}$ is bigger than $\dfrac{1}{3^n}$, but *smaller* than any other geometric series:

$$\cdots \ll \frac{1}{3^n} \ll \boxed{\frac{n}{3^n}} \ll \cdots \ll \frac{1}{(2.999)^n} \ll \frac{1}{(2.99)^n} \ll \frac{1}{(2.9)^n} \ll \frac{1}{2^n} \ll \cdots$$

EXERCISES

1–4 ■ Determine whether the given p-series converges.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

2. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}}$

3. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2}$

4. $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{\sqrt{n}}$

5–14 ■ Use the Limit Comparison Test to determine whether the given series converges or diverges.

5. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$

7. $\sum_{n=1}^{\infty} \frac{3}{4^n+5}$

8. $\sum_{n=1}^{\infty} \frac{1}{n^3+n^2}$

9. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+n!}}$

10. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

11. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+5}}$

12. $\sum_{n=1}^{\infty} \frac{5^n+1}{n^2+4^n}$

13. $\sum_{n=1}^{\infty} \frac{n^2-3n}{\sqrt[3]{n^{10}-4n^2}}$

14. $\sum_{n=1}^{\infty} \frac{2^n}{3\sqrt{5^n+n^5}}$

15–20 ■ Find the position of the given series on the hierarchy, and determine whether it converges or diverges.

15. $\sum_{n=1}^{\infty} \frac{1}{3^n\sqrt{n}}$

16. $\sum_{n=1}^{\infty} \frac{1}{(n!)^n}$

17. $\sum_{n=1}^{\infty} \frac{\ln n}{n\sqrt{n}}$

18. $\sum_{n=1}^{\infty} \frac{(\ln n)^5}{n^2}$

19. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\ln n}$

20. $\sum_{n=1}^{\infty} \frac{1}{\ln(\ln n)}$

21. Does the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converge or diverge?

22. Does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$ converge or diverge?

23. Does the series $\sum_{n=1}^{\infty} \frac{1}{(\tan^{-1} n)^n}$ converge or diverge?

24. Does the series $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$ converge or diverge?