171 Week 5 Description

Definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

1. Example: if

   $$f(x) = \frac{e^x \sin(x)}{x^2}$$

   then

   $$f'(x) = \lim_{h\to 0} \frac{\frac{e^{x+h}\sin(x+h)}{(x+h)^2} - \frac{e^x \sin(x)}{x^2}}{h}$$

   Which is too hard to do by hand, although in a week we will be able to compute this with shortcuts.

   Derivatives we know

   (a) If $f(x) = x^2$ then $f'(x) = 2x$

   (b) If $f(x) = \sqrt{x}$ then $f'(x) = \dfrac{1}{2\sqrt{x}}$

   (c) If $f(x) = x^3$ then $f'(x) = 3x^2$

   (d) More generally if $f(x) = x^n$ then $f'(x) = nx^{n-1}$

2. If

   $$f(x) = \frac{1}{x}$$

   then

   $$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

   Now some arithmetic:

   $$\frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{-h}{x(x+h)}$$

   Therefore

   $$\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{-1}{x(x+h)}$$

   Making

   $$f'(x) = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

3. Find the derivative of the function using the definition of derivative.

$$f(x) = \frac{2}{7}x + \frac{1}{4}$$

No thank you. It is a line, the slope of this line is $\frac{2}{7}$ therefore $f'(x) = \frac{2}{7}$

4. Find the derivative of the function using the definition of derivative.

$$f(x) = 8x^3 - 7x + 2$$

No again, the derivative of x^3 is $3x^2$ and the derivative of $-7x$ is -7 so

$$f'(x) = 24x^2 - 7$$

5. Find the derivative of $f(x) = \sqrt{3x+7}$, find the domain of the function and the domain of the derivative.

Let's find the domain of the function first: you cannot take the square root of a negative number, we solve

$$3x + 7 \geq 0 \iff 3x \geq -7 \iff x \geq -\frac{7}{3}$$

Webassign wants interval notation $\left[-\frac{7}{3}, \infty\right)$

Now for the derivative. A good guess would be $\frac{1}{2\sqrt{3x+7}}$ unfortunately that is wrong

$$\lim_{h \to 0} \frac{\sqrt{3(x+h)+7} - \sqrt{3x+7}}{h}$$

Same trick, multiply by top and bottom by $\sqrt{3(x+h)+7} + \sqrt{3x+7}$ get

$$\frac{3h}{h\left(\sqrt{3(x+h)+7} + \sqrt{3x+7}\right)}$$

take the limit as $h \to 0$ get $\frac{3}{2\sqrt{3x+7}}$

The domain of the derivative is almost identical to the domain of the original function but since you cannot divide by zero there is a subtle difference, it is $\left(-\frac{7}{3}, \infty\right)$ We would say "f is not differentiable at $-\frac{7}{3}$"

6. Let
$$g(x) = f^2(x)$$
then
$$g'(x) = \lim_{h\to 0} \frac{f^2(x+h) - f^2(x)}{h} = \lim_{h\to 0} \frac{(f(x+h) - f(x))(f(x+h) + f(x))}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \times \lim_{h\to 0}(f(x+h) + f(x))$$
$$= f'(x) \times (f(x) + f(x)) = 2f(x)f'(x)$$

7. If
$$g(x) = \frac{1}{f(x)}$$
follow example 2 and 4 and find
$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f^2(x)}$$

8. All mighty power rule
$$f(x) = x^n \implies f'(x) = nx^{n-1}$$

9. Example
$$f(x) = 4\sqrt{x} + x^6 \implies f'(x) = \frac{4}{2\sqrt{x}} + 6x^5$$

10. Example page 174

11.
$$f(x) = e^x \implies f'(x) = e^x$$
page 179

12. Example
$$f(x) = \frac{4x^2 + 6\sqrt{x}}{x}$$
first divide get
$$f(x) = 4x + 6x^{-\frac{1}{2}}$$
then use the almighty power rule get
$$f'(x) = 4 - 3x^{-\frac{3}{2}} = 4 - \frac{3}{\sqrt{x^3}}$$

13. Use the fact that

$$(f+g)^2 = f^2 + g^2 + 2fg$$

to solve for fg get

$$fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$$

take derivatives and deduce

$$(fg)' = f'g + g'f$$

14. Example

$$\frac{d}{dx}[\sqrt{x}e^x]$$

here we can think $f(x) = \sqrt{x}, f'(x) = \frac{1}{2\sqrt{x}}, g(x) = e^x, g'(x) = e^x$ to get

$$\frac{1}{2\sqrt{x}}e^x + \sqrt{x}e^x = e^x\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) = e^x\left(\frac{2x+1}{2\sqrt{x}}\right)$$

15. Use the "product rule" above with g replaced by $\frac{1}{g}$ to deduce the "Quotient rule"

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

See how ugly this looks in Leibniz notation page 186

16. NOT an example,

$$\frac{d}{dx}\left[\frac{3}{x^2+2x}\right]$$

use instead the rule in #5 to get

$$-\frac{3(2x+2)}{(x^2+2x)^2}$$

17. an example of the quotient rule

$$\frac{d}{dx}\left[\frac{2x+3}{x-5}\right]$$

here $f(x) = 2x+3, f'(x) = 2, g(x) = x-5, g'(x) = 1$

put it together get

$$\frac{2(x-5) - (2x+3)}{(x-5)^2} = -\frac{13}{(x-5)^2}$$

and since the derivative is negative for all values of x in the domain (why?) we know the function is always decreasing.

18. What about trig functions? Page 193

19.
$$\frac{d}{dx}\left[e^x \sin(x)\right]$$

    product rule gives

$$e^x \cos(x) + e^x \sin(x) = e^x \left(\sin(x) + \cos(x)\right)$$

20. Now try
$$f'(x) = \lim_{x \to 0} \frac{\frac{e^{x+h}\sin(x+h)}{(x+h)^2} - \frac{e^x \sin(x)}{x^2}}{h}$$

21. Table of rules page 187

22. For another table Try This