Week 4 notes on exponential models

1. This exercise uses the population growth model.

   A certain culture of the bacterium Streptococcus A initially has 11 bacteria and is observed to double every 1.5 hours. (a) Find an exponential model

   $$n(t) = n_0 2^{\frac{t}{a}}$$

   for the number of bacteria in the culture after t hours. We work with the numbers given. n_0 means how many when $t = 0$ i.e. the initial value, just like P in interest problems. a is the "doubling time": how long it take the population to double. t is the variable, standing for time. Answer is

   $$n(t) = 11 \times 2^{\frac{t}{1.5}}$$

   (b) Estimate the number of bacteria after 26 hours. (Round your answer to the nearest whole number.) Put $t = 26$ get

   $$n(26) = 11 \times 2^{\frac{26}{1.5}}$$

   wolfram Make sure it is clear how to interpret the answer. The 10^6 means move the decimal six places right.

   (c) After how many hours will the bacteria count reach 10,000? (Round your answer to one decimal place.) Solve

   $$n(t) = 11 \times 2^{\frac{t}{1.5}} = 10,000$$

   $$11 \times 2^{\frac{t}{1.5}} = 10,000 \iff 2^{\frac{t}{1.5}} = 10,000/11 \iff \frac{t}{1.5} = \frac{\ln(10,000/11)}{\ln(2)} \iff t = \frac{1.5\ln(10,000/11)}{\ln(2)}$$

   wolfram

2. This exercise uses the population growth model.

   The fox population in a certain region has a relative growth rate of 6% per year. It is estimated that the population in 2013 was 16,000. (a) Find a function

$$n(t) = n_0 e^{rt}$$

   that models the population t years after 2013.

$$n(t) = 16,000e^{.06t}$$

   (b) Use the function from part (a) to estimate the fox population in the year 2021. (Round your answer to the nearest whole number.) foxes. Put $t = 8$ get

$$n(8) = 16,000e^{.48}$$

   wolfram

   (c) After how many years will the fox population reach 19,000? (Round your answer to one decimal place.)

$$16,000e^{.06t} = 19,000 \iff e^{.06t} = 1.1875 \iff .06t = \ln(1.1875) \iff t = \frac{\ln(1.1875)}{.06}$$

   Notice this time we did not need the change of base formula because the base is e (d) Sketch a graph of the fox population function for the years 2013–2021

   wolfram

3. This exercise uses the population growth model.

   The bat population in a certain Midwestern county was 20,000 in 2017, and the observed doubling time for the population is 32 years. (a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the population t years after 2017.

   $$n(t) = 20,000 \times 2^{t/32}$$

   easy using only the numbers given.

   (b) Find an exponential model $n(t) = n_0 e^{rt}$

   This takes more work, because although the n_0 is the same, we don't know r (yet).

   To find r use the fact that if $t = 32, n(t) = 40,000$ solve

   $$20,000e^{32r} = 40,000 \iff e^{32r} = 2 \iff 32r = \ln(2) \iff r = \frac{\ln(2)}{32} \approx .02166$$

   $$n(t) = 20,000e^{.02166t}$$

   It always works the same way. If the doubling time is d years then $r = \frac{\ln(2)}{d}$

4. This exercise uses the radioactive decay model.

   The half-life of radium-226 is 1600 years. Suppose we have a 10-mg sample. (a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t years.

   $$m(t) = 10 \times 2^{-t/1600}$$

   (b) Find a function $m(t) = m_0 e^{rt}$

   that models the mass remaining after t years. (Round your r value to six decimal places.) Again we need r Since after 1600 years half is remaining we know

   $$10e^{1600r} = 5 \iff e^{1600r} = .5 \iff 1600r = \ln(.5) \iff r = \frac{\ln(.5)}{1600} \approx -0.000433$$

   $$m(t) = 10e^{-0.000433t}$$

   Again if always works this way. If the half life is h then $r = \frac{\ln(1/2)}{h}$

   (c) How much of the sample will remain after 3000 years? (Round your answer to one decimal place.) mg

   Put $t = 3500$ in either of the above?

   (d) After how many years will only 1 mg of the sample remain? (Round your answer to one decimal place.)

   $$10e^{-.000433t} = 1 \iff e^{-.000433t} = .1 \iff -.000433t = \ln(.1) \iff t = \frac{\ln(.1)}{-.000433}$$

5. This exercise uses the radioactive decay model.

   Radium-221 has a half-life of 30 sec. How long will it take for 35% of a sample to decay? (Round your answer to the nearest whole number.) First notice that you don't know how much you have to start. A good gimmick to know if you don't have a piece of information that you think you need, it doesn't matter what the number is so we can make up one. Let's say we start with 100 grams. Then 35% of it would be 35 so we would be left with 65 grams and we could solve

   $$100 \times 2^{-t/30} = 65 \iff 2^{-t/30} = .65$$

   which is actually where we should have started to begin with.

   $$2^{-t/30} = .65 \iff -t/30 = \frac{\ln(.65)}{\ln(2)} \iff t = -\frac{30\ln(.65)}{\ln(2)}$$

6. Find the exponential model containing the points $(0, 15)$ and $(2, 10)$

   This is analogous to finding the equation of the line between two points except we are looking for $y = m_0 e^{rt}$ instead of $y = mx + b$ In the line $y = mx + b, b$ is the y intercept, what you get when $x = 0$ or where the line crosses the y axis. In $y = m_0 e^{rt}$ m_0 is what you get when $t = 0$ so in this case since $(0, 15)$ is on the graph, $m_0 = 15$ Now we need r, analogous to the slope (rate of change). To find it, since $(2, 10)$ is on the curve we can replace t by 2 set the result equal to 10 and solve for r:

   $$15e^{r\times 2} = 10 \iff e^{2r} = 2/3 \iff 2r = \ln(2/3) \iff r = \frac{\ln(2/3)}{2} \approx -.203$$

   making the exponential function

   $$f(t) = 15e^{-.203t}$$

   desmos

   If you don't care about using base e we could use just the numbers given. We aren't told doubling time or half life but we are told the time it takes to get 2/3 of the original amount, and so

   $$f(t) = 10\left(\frac{2}{3}\right)^{t/2}$$

   In future work we want to use base e not base whatever.

7. You are observing bacteria in a lab. At noon there are 100. At 3:00 there are 160. Model this growth. Notice that we are not told the doubling time (increase by 100%) but we are told how long it takes to increase by 60%.

   First lets get rid of the words, and change it to "Find the exponential model that crosses through the points $(0, 100)$ and $3, 160)$ "

   Find the equation $m_0 e^{rt}$ that goes through the points $(0, 100)$ and $(3, 160)$ We know $m_0 = 100$ because $(0, 100)$ is on the curve. We need r but we know if $t = 3$ then $m(3) = 160$ so set

$$100e^{3r} = 160 \iff e^{3r} = 1.6 \iff 3r = \ln(1.6) \iff r = \frac{\ln(1.6)}{3} \approx .15667$$

$$m(t) = 100e^{.15667t}$$

   If we don't care about base e, the base will be $\frac{160}{100} = 1.6$

   The initial amount is 100 and it takes 3 hours to increase to 160 and the model is

$$m(t) = 100 \times (1.6)^{t/3}$$

8. Find the exponential model that contains the points $(0, 115)$ and $(30, 75)$ Evidently the function is decreasing so r will be negative.

$$m(t) = 115e^{rt}$$

is a start, then

$$115e^{30r} = 75 \iff e^{30r} = \frac{15}{23} \iff t = \frac{\ln(15/23)}{30} \approx -0.014248$$

$$m(t) = 115e^{-0.014248t}$$

9. This exercise uses Newton's Law of Cooling.

A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 70°F. (a) If the temperature of the turkey is 145°F after half an hour, what is its temperature after 45 min? (Round your answer to the nearest whole number.)

The temperature of the heated object (turkey) decays exponentially to the room temperature. So we work with the differences of the temperature. $185 - 70 = 115, 145 - 70 = 75$ so we are looking for the exponential model that contains the points $(0, 115)$ and $(30, 75)$

Fortunately we have the answer already, the difference between the turkeys temperature and the room temperature at time t is $115e^{-0.014248t}$

(a) What is its temperature after 45 min? (Round your answer to the nearest whole number.)

$$70 + 115e^{-.0.14248 \times 45}$$

You have to add the 70 because the model for the **differences** in the temperature. If you want the model for the actual temperature at time t it would be

$$70 + 115e^{-.014248.t}$$

After how many hours will the turkey cool to 100°F? (Round your answer to one decimal place.)

Set

$$115e^{-0.014248t} = 30$$

and solve for t

Note we could have also used

$$70 + 115\left(\frac{75}{115}\right)^{t/30}$$