Notes on logs;

$$\log_b(x) = y \iff b^y = x$$

$$\log(x) = y \iff 10^y = x$$

$$\ln(x) = y \iff e^y = x$$

Laws of Logarithms If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r\log_a x$ (where r is any real number)

The base in unimportant, this works for any base. Sometimes we will need to "expand" which means going from the left hand side to the right hand side. Other times we will need to write as a single log, going from the right to the left.
Here is the "proof" from the book:

Law 1 Let $\log_a A = u$ and $\log_a B = v$. When written in exponential form, these equations become

$$a^u = A \quad \text{and} \quad a^v = B$$

Thus

$$\begin{aligned}\log_a(AB) &= \log_a(a^u a^v) = \log_a\left(a^{a+c}\right)\\ &= u + v = \log_a A + \log_a B\end{aligned}$$

Here is a simple explanation by example: Suppose you had to multiply

$$100,000,000,000 \times 100,000,000$$

What you would do is count the zeros and add them, and therefore it would be a lot easier to have written

$$10^{11} \times 10^8$$

in other words really what we want is to add the exponents, which in this case are the logs base ten.

Since $\log(10^{11}) = 11, \log(10^8) = 8$ we see that

$$\log(10^{11} \times 10^8) = \log(10^{11}) + \log(10^8) = 11 + 8$$

If we think of logarithms as exponents, then these laws are easy to understand, as they are the laws of exponents.

1. Expand

$$\log\left(\frac{x^2y}{\sqrt{z}}\right)$$

We will take this one step by step. First by rule (2) the quotient become as subtraction

$$\log\left(\frac{x^2y}{\sqrt{z}}\right) = \log(x^2y) - \log(\sqrt{z})$$

Second, by rule (1) the product becomes a sum

$$\log(x^2y) - \log(\sqrt{z}) = \log(x^2) + \log(y) - \log(\sqrt{z})$$

Finally by rule (3) the exponents come in front as coefficients. Notice that $\sqrt{z} = z^{1/2}$ so finish with

$$\log(x^2) + \log(y) - \log(\sqrt{z}) = 2\log(x) + \log(y) - \frac{1}{2}\log(z)$$

Skipping the middle steps we see

$$\log\left(\frac{x^2y}{\sqrt{z}}\right) = 2\log(x) + \log(y) - \frac{1}{2}\log(z)$$

2. Expand

$$\log\left(\sqrt{\frac{x}{y^3z}}\right)$$

Fewer steps, but judicious use of parentheses:

$$\log\left(\sqrt{\frac{x}{y^3z}}\right) = \frac{1}{2}\left(\log(x) - (3\log(y) + \log(z))\right)$$

$$= \frac{1}{2}\log(x) - \frac{3}{2}\log(y) - \frac{1}{2}\log(z)$$

At the moment we don't really have a good reason to expand but we will later, especially in calculus where the expanded version will be much easier to deal with.

3. Combine
$$2\log(x) + 3\log(y) - \log(z)$$
In steps, by (3)
$$2\log(x) + 3\log(y) - \log(z) = \log(x^2) + \log(y^3) - \log(z)$$
By (1)
$$\log(x^2) + \log(y^3) - \log(z) = \log(x^2y^3) - \log(z)$$
Finally by (2)
$$\log(x^2y^3) - \log(z) = \log\left(\frac{x^2y^3}{z}\right)$$
or without the middle steps
$$2\log(x) + 3\log(y) - \log(z) = \log\left(\frac{x^2y^3}{z}\right)$$
Note that the answer would be the same with $2\log(x) - \log(z) + 3\log(y)$

4. Combine
$$\log(2) + 3\log(x) - \frac{1}{2}\log(x+2)$$
No steps
$$\log(2) + 3\log(x) - \frac{1}{2}\log(x+2) = \log\left(\frac{2x^3}{\sqrt{x+2}}\right)$$

5. Evaluate
$$\log_2(448) - \log_2(14)$$
By (2)
$$\log_2(488) - \log_2(14) = \log_2\left(\frac{448}{14}\right) = \log_2(32) = 5 \text{ since } 2^5 = 32$$

6. Note that (3) gives $-\log(x) = \log\left(\frac{1}{x}\right)$

   (2) gives it as well, since $\log\left(\frac{1}{x}\right) = \log(1) - \log(x) = -\log(x)$

The all mighty change of base formula :

Suppose we want to solve $2^x = 1000$ for x We can write in equivalent logarithmic form as $x = \log_2(1000)$ but that is just saying the same thing, we cannot get a numeric answer. However, starting with $2^x = 1000$ we can take the logarithm of both sides with any base we like, it doesn't have to be 2. For example we can use the natural log, or the common log

$$2^x = 1000 \iff \log(2^x) = \log(1000) \iff x\log(2) = \log(1000) \iff x = \frac{\log(1000)}{\log(2)}$$

In other words

$$x = \log_2(1000) = \frac{\log(1000)}{\log(2)}$$

More generally, we have the "change of base" formula:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)} = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}$$

Or, when solving $b^x = A$ for x we get

$$b^x = A \iff x = \frac{\log(A)}{\log(b)}$$

7. Round to three decimal places

$$\log_3(125)$$

$$\log_3(125) = \frac{\log(125)}{\log(3)} = 4.395$$

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8. Solve for t: $(1.04)^t = 2$

$$(1.04)^t = 2 \iff x = \frac{\log(2)}{\log(1.04)} = 17.637$$

wolfram

9. Invest some money at 4% interest compounded annually. How long before your money doubles? See above.