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A Modern Introduction to Probability and Statistics

Understanding Why and How

With 120 Figures

Springer

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Preface

Probability and statistics are fascinating subjects on the interface between mathematics and applied sciences that help us understand and solve practical problems. We believe that you, by learning how stochastic methods come about and why they work, will be able to understand the meaning of statistical statements as well as judge the quality of their content, when facing such problems on your own. Our philosophy is one of *how* and *why*: instead of just presenting stochastic methods as cookbook recipes, we prefer to explain the principles behind them.

In this book you will find the basics of probability theory and statistics. In addition, there are several topics that go somewhat beyond the basics but that ought to be present in an introductory course: simulation, the Poisson process, the law of large numbers, and the central limit theorem. Computers have brought many changes in statistics. In particular, the bootstrap has earned its place. It provides the possibility to derive confidence intervals and perform tests of hypotheses where traditional (normal approximation or large sample) methods are inappropriate. It is a modern useful tool one should learn about, we believe.

Examples and datasets in this book are mostly from real-life situations, at least that is what we looked for in illustrations of the material. Anybody who has inspected datasets with the purpose of using them as elementary examples knows that this is hard: on the one hand, you do not want to boldly state assumptions that are clearly not satisfied; on the other hand, long explanations concerning side issues distract from the main points. We hope that we found a good middle way.

A first course in calculus is needed as a prerequisite for this book. In addition to high-school algebra, some infinite series are used (exponential, geometric). Integration and differentiation are the most important skills, mainly concerning one variable (the exceptions, two dimensional integrals, are encountered in Chapters 9–11). Although the mathematics is kept to a minimum, we strived to be mathematically correct throughout the book. With respect to probability and statistics the book is self-contained.

The book is aimed at undergraduate engineering students, and students from more business-oriented studies (who may gloss over some of the more mathematically oriented parts). At our own university we also use it for students in applied mathematics (where we put a little more emphasis on the math and add topics like combinatorics, conditional expectations, and generating functions). It is designed for a one-semester course: on average two hours in class per chapter, the first for a lecture, the second doing exercises. The material is also well-suited for self-study, as we know from experience.

We have divided attention about evenly between probability and statistics. The very first chapter is a sampler with differently flavored introductory examples, ranging from scientific success stories to a controversial puzzle. Topics that follow are elementary probability theory, simulation, joint distributions, the law of large numbers, the central limit theorem, statistical modeling (informal: why and how we can draw inference from data), data analysis, the bootstrap, estimation, simple linear regression, confidence intervals, and hypothesis testing. Instead of a few chapters with a long list of discrete and continuous distributions, with an enumeration of the important attributes of each, we introduce a few distributions when presenting the concepts and the others where they arise (more) naturally. A list of distributions and their characteristics is found in Appendix A.

With the exception of the first one, chapters in this book consist of three main parts. First, about four sections discussing new material, interspersed with a handful of so-called Quick exercises. Working these—two-or-three-minute—exercises should help to master the material and provide a break from reading to do something more active. On about two dozen occasions you will find indented paragraphs labeled *Remark*, where we felt the need to discuss more mathematical details or background material. These remarks can be skipped without loss of continuity; in most cases they require a bit more mathematical maturity. Whenever persons are introduced in examples we have determined their sex by looking at the chapter number and applying the rule "He is odd, she is even." Solutions to the quick exercises are found in the second to last section of each chapter.

The last section of each chapter is devoted to exercises, on average thirteen per chapter. For about half of the exercises, answers are given in Appendix C, and for half of these, full solutions in Appendix D. Exercises with both a short answer and a full solution are marked with $\boxplus$ and those with only a short answer are marked with $\boxdot$ (when more appropriate, for example, in "Show that ..." exercises, the short answer provides a hint to the key step). Typically, the section starts with some easy exercises and the order of the material in the chapter is more or less respected. More challenging exercises are found at the end.

Much of the material in this book would benefit from illustration with a computer using statistical software. A complete course should also involve computer exercises. Topics like simulation, the law of large numbers, the central limit theorem, and the bootstrap loudly call for this kind of experience. For this purpose, all the datasets discussed in the book are available at http://www.springeronline.com/1-85233-896-2. The same Web site also provides access, for instructors, to a complete set of solutions to the exercises; go to the Springer online catalog or contact textbooks@springer-sbm.com to apply for your password.

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Contents

1

Why probability and statistics?

Is everything on this planet determined by randomness? This question is open to philosophical debate. What is certain is that every day thousands and thousands of engineers, scientists, business persons, manufacturers, and others are using tools from probability and statistics.

The theory and practice of probability and statistics were developed during the last century and are still actively being refined and extended. In this book we will introduce the basic notions and ideas, and in this first chapter we present a diverse collection of examples where randomness plays a role.

1.1 Biometry: iris recognition

Biometry is the art of identifying a person on the basis of his or her personal biological characteristics, such as fingerprints or voice. From recent research it appears that with the human iris one can beat all existing automatic human identification systems. Iris recognition technology is based on the visible qualities of the iris. It converts these—via a video camera—into an "iris code" consisting of just 2048 bits. This is done in such a way that the code is hardly sensitive to the size of the iris or the size of the pupil. However, at different times and different places the iris code of the same person will not be exactly the same. Thus one has to allow for a certain percentage of mismatching bits when identifying a person. In fact, the system allows about 34% mismatches! How can this lead to a reliable identification system? The miracle is that different persons have very different irides. In particular, over a large collection of different irides the code bits take the values 0 and 1 about half of the time. But that is certainly not sufficient: if one bit would determine the other 2047, then we could only distinguish two persons. In other words, single bits may be random, but the correlation between bits is also crucial (we will discuss correlation at length in Chapter 10). John Daugman who has developed the iris recognition technology made comparisons between 222 743 pairs of iris codes and concluded that of the 2048 bits 266 may be considered as uncorrelated ([6]). He then argues that we may consider an iris code as the result of 266 coin tosses with a fair coin. This implies that if we compare two such codes from different persons, then there is an astronomically small probability that these two differ in less than 34% of the bits—almost all pairs will differ in about 50% of the bits. This is illustrated in Figure 1.1, which originates from [6], and was kindly provided by John Daugman. The iris code data consist of numbers between 0 and 1, each a Hamming distance (the fraction of mismatches) between two iris codes. The data have been summarized in two histograms, that is, two graphs that show the number of counts of Hamming distances falling in a certain interval. We will encounter histograms and other summaries of data in Chapter 15. One sees from the figure that for codes from the same iris (left side) the mismatch fraction is only about 0.09, while for different irides (right side) it is about 0.46.

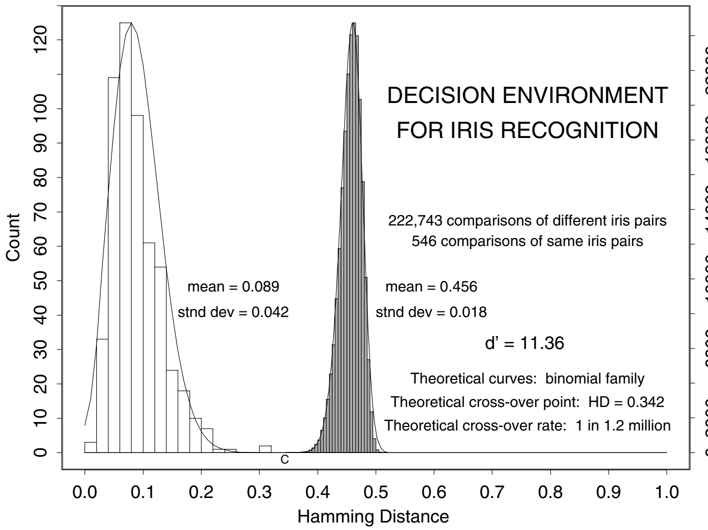


Fig. 1.1. Comparison of same and different iris pairs.

Source: J.Daugman. *Second IMA Conference on Image Processing: Mathematical Methods, Algorithms and Applications*, 2000. © Ellis Horwood Publishing Limited.

You may still wonder how it is possible that irides distinguish people so well. What about twins, for instance? The surprising thing is that although the color of eyes is hereditary, many features of iris patterns seem to be produced by so-called epigenetic events. This means that during embryo development the iris structure develops randomly. In particular, the iris patterns of (monozygotic) twins are as discrepant as those of two arbitrary individuals.

For this reason, as early as in the 1930s, eye specialists proposed that iris patterns might be used for identification purposes.

1.2 Killer football

A couple of years ago the prestigious *British Medical Journal* published a paper with the title "Cardiovascular mortality in Dutch men during 1996 European football championship: longitudinal population study" ([41]). The authors claim to have shown that the effect of a single football match is detectable in national mortality data. They consider the mortality from infarctions (heart attacks) and strokes, and the "explanation" of the increase is a combination of heavy alcohol consumption and stress caused by watching the football match on June 22 between the Netherlands and France (lost by the Dutch team!). The authors mainly support their claim with a figure like Figure 1.2, which shows the number of deaths from the causes mentioned (for men over 45), during the period June 17 to June 27, 1996. The middle horizontal line marks the average number of deaths on these days, and the upper and lower horizontal lines mark what the authors call the 95% confidence interval. The construction of such an interval is usually performed with standard statistical techniques, which you will learn in Chapter 23. The interpretation of such an interval is rather tricky. That the bar on June 22 sticks out off the confidence interval should support the "killer claim."

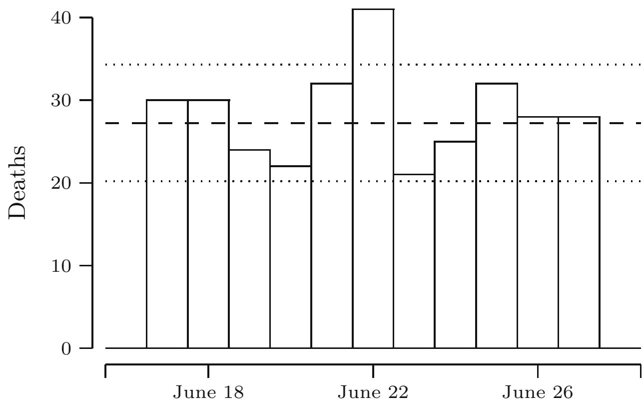


Fig. 1.2. Number of deaths from infarction or stroke in (part of) June 1996.

It is rather surprising that such a conclusion is based on a *single* football match, and one could wonder why no probability model is proposed in the paper. In fact, as we shall see in Chapter 12, it would not be a bad idea to model the time points at which deaths occur as a so-called Poisson process.

Once we have done this, we can compute how often a pattern like the one in the figure might occur—without paying attention to football matches and other high-risk national events. To do this we need the mean number of deaths per day. This number can be obtained from the data by an estimation procedure (the subject of Chapters 19 to 23). We use the sample mean, which is equal to $(10 \cdot 27.2 + 41)/11 = 313/11 = 28.45$. (Here we have to make a computation like this because we only use the data in the paper: 27.2 is the average over the 5 days preceding and following the match, and 41 is the number of deaths on the day of the match.) Now let p_{high} be the probability that there are 41 or more deaths on a day, and let p_{usual} be the probability that there are between 21 and 34 deaths on a day—here 21 and 34 are the lowest and the highest number that fall in the interval in Figure 1.2. From the formula of the Poisson distribution given in Chapter 12 one can compute that $p_{\text{high}} = 0.008$ and $p_{\text{usual}} = 0.820$. Since events on different days are independent according to the Poisson process model, the probability p of a pattern as in the figure is

$$p = p_{\text{usual}}^5 \cdot p_{\text{high}} \cdot p_{\text{usual}}^5 = 0.0011.$$

From this it can be shown by (a generalization of) the law of large numbers (which we will study in Chapter 13) that such a pattern would appear about once every $1/0.0011 = 899$ days. So it is not overwhelmingly exceptional to find such a pattern, and the fact that there was an important football match on the day in the middle of the pattern might just have been a coincidence.

1.3 Cars and goats: the Monty Hall dilemma

On Sunday September 9, 1990, the following question appeared in the "Ask Marilyn" column in *Parade*, a Sunday supplement to many newspapers across the United States:

> Suppose you're on a game show, and you're given the choice of three doors; behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?—Craig F. Whitaker, Columbia, Md.

Marilyn's answer—one should switch—caused an avalanche of reactions, in total an estimated 10 000. Some of these reactions were not so flattering ("You are the goat"), quite a lot were by professional mathematicians ("You blew it, and blew it big," "You are utterly incorrect .... How many irate mathematicians are needed to change your mind?"). Perhaps some of the reactions were so strong, because Marilyn vos Savant, the author of the column, is in the *Guinness Book of Records* for having one of the highest IQs in the world.

The switching question was inspired by Monty Hall's "Let's Make a Deal" game show, which ran with small interruptions for 23 years on various U.S. television networks.

Although it is not explicitly stated in the question, the game show host will *always* open a door with a goat after you make your initial choice. Many people would argue that in this situation it does not matter whether one would change or not: one door has a car behind it, the other a goat, so the odds to get the car are fifty-fifty. To see why they are wrong, consider the following argument. In the original situation two of the three doors have a goat behind them, so with probability 2/3 your initial choice was wrong, and with probability 1/3 it was right. Now the host opens a door with a goat (note that he can always do this). In case your initial choice was *wrong* the host has only one option to show a door with a goat, and switching leads you to the door with the car. In case your initial choice was *right* the host has two goats to choose from, so switching will lead you to a goat. We see that switching is the best strategy, doubling our chances to win. To stress this argument, consider the following generalization of the problem: suppose there are 10 000 doors, behind one is a car and behind the rest, goats. After you make your choice, the host will open 9998 doors with goats, and offers you the option to switch. To change or not to change, that's the question! Still not convinced? Use your Internet browser to find one of the zillion sites where one can run a simulation of the Monty Hall problem (more about simulation in Chapter 6).

In fact, there are quite a lot of variations on the problem. For example, the situation that there are four doors: you select a door, the host always opens a door with a goat, and offers you to select another door. After you have made up your mind he opens a door with a goat, and again offers you to switch. After you have decided, he opens the door you selected. What is now the best strategy? In this situation switching only at the last possible moment yields a probability of 3/4 to bring the car home. Using the law of total probability from Section 3.3 you will find that this is indeed the best possible strategy.

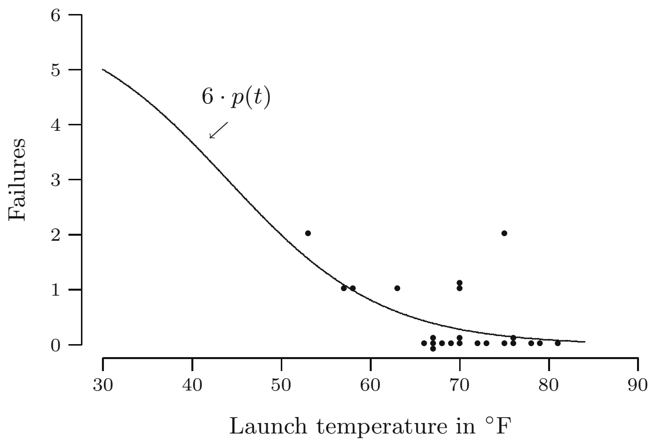
1.4 The space shuttle *Challenger*

On January 28, 1986, the space shuttle *Challenger* exploded about one minute after it had taken off from the launch pad at Kennedy Space Center in Florida. The seven astronauts on board were killed and the spacecraft was destroyed. The cause of the disaster was explosion of the main fuel tank, caused by flames of hot gas erupting from one of the so-called solid rocket boosters.

These solid rocket boosters had been cause for concern since the early years of the shuttle. They are manufactured in segments, which are joined at a later stage, resulting in a number of joints that are sealed to protect against leakage. This is done with so-called O-rings, which in turn are protected by a layer of putty. When the rocket motor ignites, high pressure and high temperature build up within. In time these may burn away the putty and subsequently erode the O-rings, eventually causing hot flames to erupt on the outside. In a nutshell, this is what actually happened to the *Challenger*.

After the explosion, an investigative commission determined the causes of the disaster, and a report was issued with many findings and recommendations ([24]). On the evening of January 27, a decision to launch the next day had been made, notwithstanding the fact that an extremely low temperature of 31°F had been predicted, well below the operating limit of 40°F set by Morton Thiokol, the manufacturer of the solid rocket boosters. Apparently, a "management decision" was made to overrule the engineers' recommendation not to launch. The inquiry faulted both NASA and Morton Thiokol management for giving in to the pressure to launch, ignoring warnings about problems with the seals.

The *Challenger* launch was the 24th of the space shuttle program, and we shall look at the data on the number of failed O-rings, available from previous launches (see [5] for more details). Each rocket has three O-rings, and two rocket boosters are used per launch, so in total six O-rings are used each time. Because low temperatures are known to adversely affect the O-rings, we also look at the corresponding launch temperature. In Figure 1.3 the dots show the number of failed O-rings per mission (there are 23 dots—one time the boosters could not be recovered from the ocean; temperatures are rounded to the nearest degree Fahrenheit; in case of two or more equal data points these are shifted slightly.). If you ignore the dots representing zero failures, which all occurred at high temperatures, a temperature effect is not apparent.



Source: based on data from Volume VI of the Report of the Presidential Commission on the space shuttle Challenger accident, Washington, DC, 1986.

Fig. 1.3. Space shuttle failure data of pre-*Challenger* missions and fitted model of expected number of failures per mission function.

In a model to describe these data, the probability $p(t)$ that an individual O-ring fails should depend on the launch temperature t. Per mission, the number of failed O-rings follows a so-called binomial distribution: six O-rings, and each may fail with probability $p(t)$; more about this distribution and the circumstances under which it arises can be found in Chapter 4. A *logistic* model was used in [5] to describe the dependence on t:

$$p(t) = \frac{\mathrm{e}^{a+b\cdot t}}{1 + \mathrm{e}^{a+b\cdot t}}.$$

A high value of $a + b \cdot t$ corresponds to a high value of $p(t)$, a low value to low $p(t)$. Values of a and b were determined from the data, according to the following principle: choose a and b so that the probability that we get data as in Figure 1.3 is as high as possible. This is an example of the use of the method of maximum likelihood, which we shall discuss in Chapter 21. This results in $a = 5.085$ and $b = -0.1156$, which indeed leads to lower probabilities at higher temperatures, and to $p(31) = 0.8178$. We can also compute the (estimated) expected number of failures, $6 \cdot p(t)$, as a function of the launch temperature t; this is the plotted line in the figure.

Combining the estimates with estimated probabilities of other events that should happen for a *complete* failure of the field-joint, the estimated probability of such a failure is 0.023. With six field-joints, the probability of at least one complete failure is then $1 - (1 - 0.023)^6 = 0.13$!

1.5 Statistics versus intelligence agencies

During World War II, information about Germany's war potential was essential to the Allied forces in order to schedule the time of invasions and to carry out the allied strategic bombing program. Methods for estimating German production used during the early phases of the war proved to be inadequate. In order to obtain more reliable estimates of German war production, experts from the Economic Warfare Division of the American Embassy and the British Ministry of Economic Warfare started to analyze markings and serial numbers obtained from captured German equipment.

Each piece of enemy equipment was labeled with markings, which included all or some portion of the following information: (a) the name and location of the marker; (b) the date of manufacture; (c) a serial number; and (d) miscellaneous markings such as trademarks, mold numbers, casting numbers, etc. The purpose of these markings was to maintain an effective check on production standards and to perform spare parts control. However, these same markings offered Allied intelligence a wealth of information about German industry.

The first products to be analyzed were tires taken from German aircraft shot over Britain and from supply dumps of aircraft and motor vehicle tires captured in North Africa. The marking on each tire contained the maker's name, a serial number, and a two-letter code for the date of manufacture. The first step in analyzing the tire markings involved breaking the two-letter date code. It was conjectured that one letter represented the month and the other the year of manufacture, and that there should be 12 letter variations for the month code and 3 to 6 for the year code. This, indeed, turned out to be true. The following table presents examples of the 12 letter variations used by four different manufacturers.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Dunlop	T	I	E	B	R	A	P	O	L	N	U	D
Fulda	F	U	L	D	A	M	U	N	S	T	E	R
Phoenix	F	O	N	I	X	H	A	M	B	U	R	G
Sempirit	A	B	C	D	E	F	G	H	I	J	K	L

For instance, the Dunlop code was Dunlop Arbeit spelled backwards. Next, the year code was broken and the numbering system was solved so that for each manufacturer individually the serial numbers could be dated. Moreover, for each month, the serial numbers could be recoded to numbers running from 1 to some unknown largest number N, and the observed (recoded) serial numbers could be seen as a subset of this. The objective was to estimate N for each month and each manufacturer separately by means of the observed (recoded) serial numbers. In Chapter 20 we discuss two different methods of estimation, and we show that the method based on only the maximum observed (recoded) serial number is much better than the method based on the average observed (recoded) serial numbers.

With a sample of about 1400 tires from five producers, individual monthly output figures were obtained for almost all months over a period from 1939 to mid-1943. The following table compares the accuracy of estimates of the average monthly production of all manufacturers of the first quarter of 1943 with the statistics of the Speer Ministry that became available after the war. The accuracy of the estimates can be appreciated even more if we compare them with the figures obtained by Allied intelligence agencies. They estimated, using other methods, the production between 900 000 and 1 200 000 per month!

Type of tire	Estimated production	Actual production
Truck and passenger car	147 000	159 000
Aircraft	28 500	26 400
Total	175 500	186 100

1.6 The speed of light

In 1983 the definition of the meter (the SI unit of one meter) was changed to: *The meter is the length of the path traveled by light in vacuum during a time interval of* 1/299 792 458 *of a second.* This implicitly defines the speed of light as 299 792 458 meters per second. It was done because one thought that the speed of light was so accurately known that it made more sense to define the meter in terms of the speed of light rather than vice versa, a remarkable end to a long story of scientific discovery. For a long time most scientists believed that the speed of light was infinite. Early experiments devised to demonstrate the finiteness of the speed of light failed because the speed is so extraordinarily high. In the 18th century this debate was settled, and work started on determination of the speed, using astronomical observations, but a century later scientists turned to earth-based experiments. Albert Michelson refined experimental arrangements from two previous experiments and conducted a series of measurements in June and early July of 1879, at the U.S. Naval Academy in Annapolis. In this section we give a very short summary of his work. It is extracted from an article in *Statistical Science* ([18]).

The principle of speed measurement is easy, of course: measure a distance and the time it takes to travel that distance, the speed equals distance divided by time. For an accurate determination, both the distance and the time need to be measured accurately, and with the speed of light this is a problem: either we should use a very large distance and the accuracy of the distance measurement is a problem, or we have a very short time interval, which is also very difficult to measure accurately.

In Michelson's time it was known that the speed of light was about 300 000 km/s, and he embarked on his study with the goal of an improved value of the speed of light. His experimental setup is depicted schematically in Figure 1.4. Light emitted from a light source is aimed, through a slit in a fixed plate, at a rotating mirror; we call its distance from the plate the radius. At one particular angle, this rotating mirror reflects the beam in the direction of a distant (fixed) flat mirror. On its way the light first passes through a focusing lens. This second mirror is positioned in such a way that it reflects the beam back in the direction of the rotating mirror. In the time it takes the light to travel back and forth between the two mirrors, the rotating mirror has moved by an angle α, resulting in a reflection on the plate that is displaced with respect to the source beam that passed through the slit. The radius and the displacement determine the angle α because

$$\tan 2\alpha = \frac{\text{displacement}}{\text{radius}}$$

and combined with the number of revolutions per seconds (rps) of the mirror, this determines the elapsed time:

$$\text{time} = \frac{\alpha/2\pi}{\text{rps}}.$$

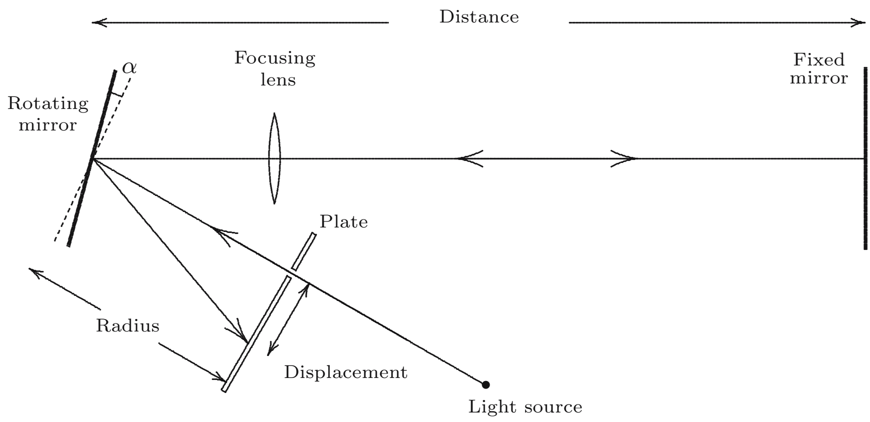


Fig. 1.4. Michelson's experiment.

During this time the light traveled twice the distance between the mirrors, so the speed of light in air now follows:

$$c_{\text{air}} = \frac{2 \cdot \text{distance}}{\text{time}}.$$

All in all, it looks simple: just measure the four quantities—distance, radius, displacement and the revolutions per second—and do the calculations. This is much harder than it looks, and problems in the form of inaccuracies are lurking everywhere. An error in any of these quantities translates directly into some error in the final result.

Michelson did the utmost to reduce errors. For example, the distance between the mirrors was about 2000 feet, and to measure it he used a steel measuring tape. Its nominal length was 100 feet, but he carefully checked this using a copy of the official "standard yard." He found that the tape was in fact 100.006 feet. This way he eliminated a (small) systematic error.

Now imagine using the tape to measure a distance of 2000 feet: you have to use the tape 20 times, each time marking the next 100 feet. Do it again, and you probably find a slightly different answer, no matter how hard you try to be very precise in every step of the measuring procedure. This kind of variation is inevitable: sometimes we end up with a value that is a bit too high, other times it is too low, but on average we're doing okay—assuming that we have eliminated sources of systematic error, as in the measuring tape. Michelson measured the distance five times, which resulted in values between 1984.93 and 1985.17 feet (after correcting for the temperature-dependent stretch), and he used the average as the "true distance."

In many phases of the measuring process Michelson attempted to identify and determine systematic errors and subsequently applied corrections. He also systematically repeated measuring steps and averaged the results to reduce variability. His final dataset consists of 100 separate measurements (see Table 17.1), but each is in fact summarized and averaged from repeated measurements on several variables. The final result he reported was that the speed of light in vacuum (this involved a conversion) was $299\,944 \pm 51$ km/s, where the 51 is an indication of the uncertainty in the answer. In retrospect, we must conclude that, in spite of Michelson's admirable meticulousness, some source of error must have slipped his attention, as his result is off by about 150 km/s. With current methods we would derive from his data a so-called 95% confidence interval: $299\,944 \pm 15.5$ km/s, suggesting that Michelson's uncertainty analysis was a little conservative. The methods used to construct confidence intervals are the topic of Chapters 23 and 24.

2

Outcomes, events, and probability

The world around us is full of phenomena we perceive as random or unpredictable. We aim to model these phenomena as *outcomes* of some experiment, where you should think of *experiment* in a very general sense. The outcomes are elements of a *sample space* Ω, and subsets of Ω are called *events*.The events will be assigned a *probability*, a number between 0 and 1 that expresses how likely the event is to occur.

2.1 Sample spaces

Sample spaces are simply sets whose elements describe the outcomes of the experiment in which we are interested.

We start with the most basic experiment: the tossing of a coin. Assuming that we will never see the coin land on its rim, there are two possible outcomes: heads and tails. We therefore take as the sample space associated with this experiment the set $\Omega = \{H, T\}$.

In another experiment we ask the next person we meet on the street in which month her birthday falls. An obvious choice for the sample space is

$$\Omega = \{\text{Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sep, Oct, Nov, Dec}\}.$$

In a third experiment we load a scale model for a bridge up to the point where the structure collapses. The outcome is the load at which this occurs. In reality, one can only measure with finite accuracy, e.g., to five decimals, and a sample space with just those numbers would strictly be adequate. However, in principle, the load itself could be any positive number and therefore $\Omega = (0, \infty)$ is the right choice. Even though in reality there may also be an upper limit to what loads are conceivable, it is not necessary or practical to try to limit the outcomes correspondingly.

In a fourth experiment, we find on our doormat three envelopes, sent to us by three different persons, and we look in which order the envelopes lie on top of each other. Coding them 1, 2, and 3, the sample space would be

$$\Omega = \{123, 132, 213, 231, 312, 321\}.$$

QUICK EXERCISE 2.1 If we received mail from four different persons, how many elements would the corresponding sample space have?

In general one might consider the order in which n different objects can be placed. This is called a *permutation* of the n objects. As we have seen, there are 6 possible permutations of 3 objects, and $4 \cdot 6 = 24$ of 4 objects. What happens is that if we add the nth object, then this can be placed in any of n positions in any of the permutations of $n-1$ objects. Therefore there are

$$n \cdot (n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 = n!$$

possible permutations of n objects. Here $n!$ is the standard notation for this product and is pronounced "n factorial." It is convenient to define $0! = 1$.

2.2 Events

Subsets of the sample space are called *events*. We say that an event A *occurs* if the outcome of the experiment is an element of the set A. For example, in the birthday experiment we can ask for the outcomes that correspond to a long month, i.e., a month with 31 days. This is the event

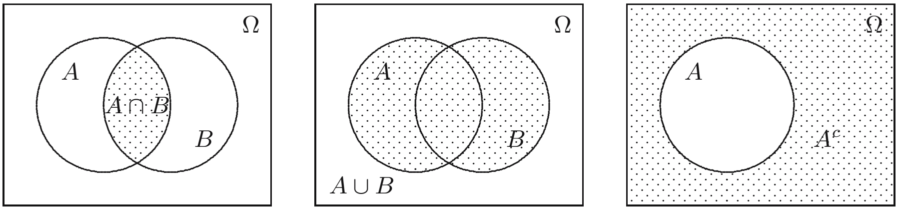
$$L = \{\text{Jan, Mar, May, Jul, Aug, Oct, Dec}\}.$$

Events may be combined according to the usual set operations.

For example if R is the event that corresponds to the months that have the letter r in their (full) name (so $R = \{\text{Jan}, \text{Feb}, \text{Mar}, \text{Apr}, \text{Sep}, \text{Oct}, \text{Nov}, \text{Dec}\}$), then the long months that contain the letter r are

$$L \cap R = \{\text{Jan}, \text{Mar}, \text{Oct}, \text{Dec}\}.$$

The set $L \cap R$ is called the *intersection* of L and R and occurs if both L and R occur. Similarly, we have the *union* $A \cup B$ of two sets A and B, which occurs if at least one of the events A and B occurs. Another common operation is taking complements. The event $A^c = \{\omega \in \Omega : \omega \notin A\}$ is called the *complement* of A; it occurs if and only if A does *not* occur. The complement of Ω is denoted $\emptyset$, the empty set, which represents the impossible event. Figure 2.1 illustrates these three set operations.



Intersection $A \cap B$ Union $A \cup B$ Complement A^c

Fig. 2.1. Diagrams of intersection, union, and complement.

We call events A and B *disjoint* or *mutually exclusive* if A and B have no outcomes in common; in set terminology: $A \cap B = \emptyset$. For example, the event L "the birthday falls in a long month" and the event {Feb} are disjoint.

Finally, we say that event A *implies* event B if the outcomes of A also lie in B. In set notation: $A \subset B$; see Figure 2.2.

Some people like to use double negations:

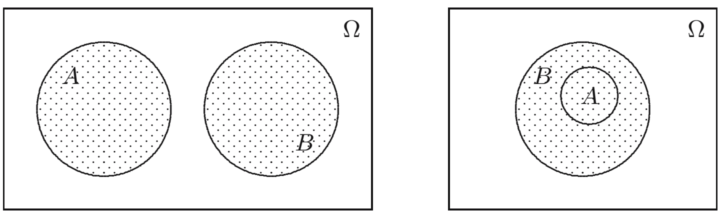
"It is certainly not true that neither John nor Mary is to blame."

This is equivalent to: "John or Mary is to blame, or both." The following useful rules formalize this mental operation to a manipulation with events.

DEMORGAN'S LAWS. For any two events A and B we have

$$(A \cup B)^c = A^c \cap B^c \text{ and } (A \cap B)^c = A^c \cup B^c.$$

QUICK EXERCISE 2.2 Let J be the event "John is to blame" and M the event "Mary is to blame." Express the two statements above in terms of the events J, J^c, M, and M^c, and check the equivalence of the statements by means of DeMorgan's laws.



Disjoint sets A and B A subset of B

Fig. 2.2. Minimal and maximal intersection of two sets.

2.3 Probability

We want to express how likely it is that an event occurs. To do this we will assign a probability to each event. The assignment of probabilities to events is in general not an easy task, and some of the coming chapters will be dedicated directly or indirectly to this problem. Since *each* event has to be assigned a probability, we speak of a probability *function*. It has to satisfy two basic properties.

> DEFINITION. A *probability function* P on a finite sample space Ω assigns to each event A in Ω a number $\mathrm{P}(A)$ in [0,1] such that
> (i) $\mathrm{P}(\Omega) = 1$, and
> (ii) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ if A and B are disjoint.
> The number $\mathrm{P}(A)$ is called the probability that A occurs.

Property (i) expresses that the outcome of the experiment is always an element of the sample space, and property (ii) is the additivity property of a probability function. It implies additivity of the probability function over more than two sets; e.g., if A, B, and C are disjoint events, then the two events $A \cup B$ and C are also disjoint, so

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A \cup B) + \mathrm{P}(C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C)\,.$$

We will now look at some examples. When we want to decide whether Peter or Paul has to wash the dishes, we might toss a coin. The fact that we consider this a fair way to decide translates into the opinion that heads and tails are equally likely to occur as the outcome of the coin-tossing experiment. So we put

$$\mathrm{P}(\{H\}) = \mathrm{P}(\{T\}) = \frac{1}{2}.$$

Formally we have to write $\{H\}$ for the set consisting of the single element H, because a probability function is defined on *events*, not on outcomes. From now on we shall drop these brackets.

Now it might happen, for example due to an asymmetric distribution of the mass over the coin, that the coin is not completely fair. For example, it might be the case that

$$\mathrm{P}(H) = 0.4999 \text{ and } \mathrm{P}(T) = 0.5001.$$

More generally we can consider experiments with two possible outcomes, say "failure" and "success", which have probabilities $1 - p$ and p to occur, where p is a number between 0 and 1. For example, when our experiment consists of buying a ticket in a lottery with 10 000 tickets and only one prize, where "success" stands for winning the prize, then $p = 10^{-4}$.

How should we assign probabilities in the second experiment, where we ask for the month in which the next person we meet has his or her birthday? In analogy with what we have just done, we put

$$\mathrm{P}(\mathrm{Jan}) = \mathrm{P}(\mathrm{Feb}) = \cdots = \mathrm{P}(\mathrm{Dec}) = \frac{1}{12}.$$

Some of you might object to this and propose that we put, for example,

$$\mathrm{P}(\mathrm{Jan}) = \frac{31}{365} \quad \text{and} \quad \mathrm{P}(\mathrm{Apr}) = \frac{30}{365},$$

because we have long months and short months. But then the very precise among us might remark that this does not yet take care of leap years.

Quick exercise 2.3 If you would take care of the leap years, assuming that one in every four years is a leap year (which again is an approximation to reality!), how would you assign a probability to each month?

In the third experiment (the buckling load of a bridge), where the outcomes are real numbers, it is impossible to assign a positive probability to each outcome (there are just too many outcomes!). We shall come back to this problem in Chapter 5, restricting ourselves in this chapter to finite and countably infinite[1] sample spaces.

In the fourth experiment it makes sense to assign equal probabilities to all six outcomes:

$$\mathrm{P}(123) = \mathrm{P}(132) = \mathrm{P}(213) = \mathrm{P}(231) = \mathrm{P}(312) = \mathrm{P}(321) = \frac{1}{6}.$$

Until now we have only assigned probabilities to the individual outcomes of the experiments. To assign probabilities to events we use the additivity property. For instance, to find the probability $\mathrm{P}(T)$ of the event T that in the three envelopes experiment envelope 2 is on top we note that

$$\mathrm{P}(T) = \mathrm{P}(213) + \mathrm{P}(231) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

In general, additivity of P implies that the probability of an event is obtained by summing the probabilities of the outcomes belonging to the event.

Quick exercise 2.4 Compute $\mathrm{P}(L)$ and $\mathrm{P}(R)$ in the birthday experiment.

Finally we mention a rule that permits us to compute probabilities of events A and B that are *not* disjoint. Note that we can write $A = (A \cap B) \cup (A \cap B^c)$, which is a disjoint union; hence

$$\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^c)\,.$$

If we split $A \cup B$ in the same way with B and B^c, we obtain the events $(A \cup B) \cap B$, which is simply B and $(A \cup B) \cap B^c$, which is nothing but $A \cap B^c$.

[1] This means: although infinite, we can still count them one by one; $\Omega = \{\omega_1, \omega_2, \ldots\}$. The interval [0,1] of real numbers is an example of an uncountable sample space.

Thus

$$\mathrm{P}(A \cup B) = \mathrm{P}(B) + \mathrm{P}(A \cap B^c)\,.$$

Eliminating $\mathrm{P}(A \cap B^c)$ from these two equations we obtain the following rule.

The probability of a union. For any two events A and B we have

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)\,.$$

From the additivity property we can also find a way to compute probabilities of complements of events: from $A \cup A^c = \Omega$, we deduce that

$$\mathrm{P}(A^c) = 1 - \mathrm{P}(A)\,.$$

2.4 Products of sample spaces

Basic to statistics is that one usually does not consider *one* experiment, but that the same experiment is performed several times. For example, suppose we throw a coin two times. What is the sample space associated with this new experiment? It is clear that it should be the set

$$\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}.$$

If in the original experiment we had a fair coin, i.e., $\mathrm{P}(H) = \mathrm{P}(T)$, then in this new experiment all 4 outcomes again have equal probabilities:

$$\mathrm{P}((H, H)) = \mathrm{P}((H, T)) = \mathrm{P}((T, H)) = \mathrm{P}((T, T)) = \frac{1}{4}.$$

Somewhat more generally, if we consider two experiments with sample spaces Ω_1 and Ω_2 then the combined experiment has as its sample space the set

$$\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}.$$

If Ω_1 has r elements and Ω_2 has s elements, then $\Omega_1 \times \Omega_2$ has rs elements. Now suppose that in the first, the second, and the combined experiment all outcomes are equally likely to occur. Then the outcomes in the first experiment have probability $1/r$ to occur, those of the second experiment $1/s$, and those of the combined experiment probability $1/rs$. Motivated by the fact that $1/rs = (1/r) \times (1/s)$, we will assign probability $p_i p_j$ to the outcome (ω_i, ω_j) in the combined experiment, in the case that ω_i has probability p_i and ω_j has probability p_j to occur. One should realize that this is by no means the only way to assign probabilities to the outcomes of a combined experiment. The preceding choice corresponds to the situation where the two experiments do not influence each other in any way. What we mean by this influence will be explained in more detail in the next chapter.

Quick exercise 2.5 Consider the sample space $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ of some experiment, where outcome a_i has probability p_i for $i = 1, \ldots, 6$. We perform this experiment twice in such a way that the associated probabilities are

$$\mathrm{P}((a_i, a_i)) = p_i, \quad \text{and} \quad \mathrm{P}((a_i, a_j)) = 0 \;\text{ if }\; i \neq j, \quad \text{for } i, j = 1, \ldots, 6.$$

Check that P is a probability function on the sample space $\Omega = \{a_1, \ldots, a_6\} \times \{a_1, \ldots, a_6\}$ of the combined experiment. What is the relationship between the first experiment and the second experiment that is determined by this probability function?

We started this section with the experiment of throwing a coin twice. If we want to learn more about the randomness associated with a particular experiment, then we should repeat it more often, say n times. For example, if we perform an experiment with outcomes 1 (success) and 0 (failure) five times, and we consider the event A "exactly one experiment was a success," then this event is given by the set

$$A = \{(0,0,0,0,1), (0,0,0,1,0), (0,0,1,0,0), (0,1,0,0,0), (1,0,0,0,0)\}$$

in $\Omega = \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}$. Moreover, if success has probability p and failure probability $1 - p$, then

$$\mathrm{P}(A) = 5 \cdot (1-p)^4 \cdot p,$$

since there are five outcomes in the event A, each having probability $(1-p)^4 \cdot p$.

Quick exercise 2.6 What is the probability of the event B "exactly two experiments were successful"?

In general, when we perform an experiment n times, then the corresponding sample space is

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n,$$

where Ω_i for $i = 1, \ldots, n$ is a copy of the sample space of the original experiment. Moreover, we assign probabilities to the outcomes $(\omega_1, \ldots, \omega_n)$ in the standard way described earlier, i.e.,

$$\mathrm{P}((\omega_1, \omega_2, \ldots, \omega_n)) = p_1 \cdot p_2 \cdots p_n,$$

if each ω_i has probability p_i.

2.5 An infinite sample space

We end this chapter with an example of an experiment with infinitely many outcomes. We toss a coin repeatedly until the first head turns up. The outcome of the experiment is the number of tosses it takes to have this first occurrence of a head. Our sample space is the space of all positive natural numbers

$$\Omega = \{1, 2, 3, \ldots\}.$$

What is the probability function P for this experiment?

Suppose the coin has probability p of falling on heads and probability $1-p$ to fall on tails, where $0 < p < 1$. We determine the probability $\mathrm{P}(n)$ for each n. Clearly $\mathrm{P}(1) = p$, the probability that we have a head right away. The event $\{2\}$ corresponds to the outcome (T, H) in $\{H, T\} \times \{H, T\}$, so we should have

$$\mathrm{P}(2) = (1-p)p.$$

Similarly, the event $\{n\}$ corresponds to the outcome $(T, T, \ldots, T, T, H)$ in the space $\{H, T\} \times \cdots \times \{H, T\}$. Hence we should have, in general,

$$\mathrm{P}(n) = (1-p)^{n-1}p, \qquad n = 1, 2, 3, \ldots.$$

Does this define a probability function on $\Omega = \{1, 2, 3, \ldots\}$? Then we should at least have $\mathrm{P}(\Omega) = 1$. It is not directly clear how to calculate $\mathrm{P}(\Omega)$: since the sample space is no longer finite we have to amend the definition of a probability function.

> DEFINITION. A *probability function* on an infinite (or finite) sample space Ω assigns to each event A in Ω a number $\mathrm{P}(A)$ in $[0, 1]$ such that
> (i) $\mathrm{P}(\Omega) = 1$, and
> (ii) $\mathrm{P}(A_1 \cup A_2 \cup A_3 \cup \cdots) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \mathrm{P}(A_3) + \cdots$
> if $A_1, A_2, A_3, \ldots$ are disjoint events.

Note that this new additivity property is an extension of the previous one because if we choose $A_3 = A_4 = \cdots = \emptyset$, then

$$\begin{aligned}
\mathrm{P}(A_1 \cup A_2) &= \mathrm{P}(A_1 \cup A_2 \cup \emptyset \cup \emptyset \cup \cdots) \\
&= \mathrm{P}(A_1) + \mathrm{P}(A_2) + 0 + 0 + \cdots = \mathrm{P}(A_1) + \mathrm{P}(A_2)\,.
\end{aligned}$$

Now we can compute the probability of Ω:

$$\begin{aligned}
\mathrm{P}(\Omega) &= \mathrm{P}(1) + \mathrm{P}(2) + \cdots + \mathrm{P}(n) + \cdots \\
&= p + (1-p)p + \cdots (1-p)^{n-1}p + \cdots \\
&= p[1 + (1-p) + \cdots (1-p)^{n-1} + \cdots].
\end{aligned}$$

The sum $1 + (1-p) + \cdots + (1-p)^{n-1} + \cdots$ is an example of a *geometric series*. It is well known that when $|1-p| < 1$,

$$1 + (1-p) + \cdots + (1-p)^{n-1} + \cdots = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

Therefore we do indeed have $\mathrm{P}(\Omega) = p \cdot \dfrac{1}{p} = 1$.

Quick exercise 2.7 Suppose an experiment in a laboratory is repeated every day of the week until it is successful, the probability of success being p. The first experiment is started on a Monday. What is the probability that the series ends on the next Sunday?

2.6 Solutions to the quick exercises

2.1 The sample space is $\Omega = \{1234, 1243, 1324, 1342, \ldots, 4321\}$. The best way to count its elements is by noting that for *each* of the 6 outcomes of the three-envelope experiment we can put a fourth envelope in any of 4 positions. Hence Ω has $4 \cdot 6 = 24$ elements.

2.2 The statement "It is certainly not true that neither John nor Mary is to blame" corresponds to the event $(J^c \cap M^c)^c$. The statement "John or Mary is to blame, or both" corresponds to the event $J \cup M$. Equivalence now follows from DeMorgan's laws.

2.3 In four years we have $365 \times 3 + 366 = 1461$ days. Hence long months each have a probability $4 \times 31/1461 = 124/1461$, and short months a probability $120/1461$ to occur. Moreover, $\{\text{Feb}\}$ has probability $113/1461$.

2.4 Since there are 7 long months and 8 months with an "r" in their name, we have $\mathrm{P}(L) = 7/12$ and $\mathrm{P}(R) = 8/12$.

2.5 Checking that P is a probability function Ω amounts to verifying that $0 \leq \mathrm{P}((a_i, a_j)) \leq 1$ for all i and j and noting that

$$\mathrm{P}(\Omega) = \sum_{i,j=1}^{6} \mathrm{P}((a_i, a_j)) = \sum_{i=1}^{6} \mathrm{P}((a_i, a_i)) = \sum_{i=1}^{6} p_i = 1.$$

The two experiments are *totally* coupled: one has outcome a_i if and only if the other has outcome a_i.

2.6 Now there are 10 outcomes in B (for example (0,1,0,1,0)), each having probability $(1-p)^3 p^2$. Hence $\mathrm{P}(B) = 10(1-p)^3 p^2$.

2.7 This happens if and only if the experiment fails on Monday,..., Saturday, and is a success on Sunday. This has probability $p(1-p)^6$ to happen.

2.7 Exercises

2.1 ⊡ Let A and B be two events in a sample space for which $\mathrm{P}(A) = 2/3$, $\mathrm{P}(B) = 1/6$, and $\mathrm{P}(A \cap B) = 1/9$. What is $\mathrm{P}(A \cup B)$?

2.2 Let E and F be two events for which one knows that the probability that at least one of them occurs is 3/4. What is the probability that neither E nor F occurs? *Hint:* use one of DeMorgan's laws: $E^c \cap F^c = (E \cup F)^c$.

2.3 Let C and D be two events for which one knows that $\mathrm{P}(C) = 0.3, \mathrm{P}(D) = 0.4$, and $\mathrm{P}(C \cap D) = 0.2$. What is $\mathrm{P}(C^c \cap D)$?

2.4 ⊡ We consider events A, B, and C, which can occur in some experiment. Is it true that the probability that *only* A occurs (and not B or C) is equal to $\mathrm{P}(A \cup B \cup C) - \mathrm{P}(B) - \mathrm{P}(C) + \mathrm{P}(B \cap C)$?

2.5 The event $A \cap B^c$ that A occurs but not B is sometimes denoted as $A \setminus B$. Here $\setminus$ is the set-theoretic minus sign. Show that $\mathrm{P}(A \setminus B) = \mathrm{P}(A) - \mathrm{P}(B)$ if B implies A, i.e., if $B \subset A$.

2.6 When $\mathrm{P}(A) = 1/3$, $\mathrm{P}(B) = 1/2$, and $\mathrm{P}(A \cup B) = 3/4$, what is

a. $\mathrm{P}(A \cap B)$?
b. $\mathrm{P}(A^c \cup B^c)$?

2.7 ⊡ Let A and B be two events. Suppose that $\mathrm{P}(A) = 0.4$, $\mathrm{P}(B) = 0.5$, and $\mathrm{P}(A \cap B) = 0.1$. Find the probability that A or B occurs, but not both.

2.8 ⊞ Suppose the events D_1 and D_2 represent disasters, which are rare: $\mathrm{P}(D_1) \leq 10^{-6}$ *and* $\mathrm{P}(D_2) \leq 10^{-6}$. What can you say about the probability that at least one of the disasters occurs? What about the probability that they *both* occur?

2.9 We toss a coin three times. For this experiment we choose the sample space

$$\Omega = \{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$$

where T stands for tails and H for heads.

a. Write down the set of outcomes corresponding to each of the following events:

A : "we throw tails exactly two times."
B : "we throw tails at least two times."
C : "tails did not appear *before* a head appeared."
D : "the first throw results in tails."

b. Write down the set of outcomes corresponding to each of the following events: A^c, $A \cup (C \cap D)$, and $A \cap D^c$.

2.10 In some sample space we consider two events A and B. Let C be the event that A or B occurs, but not both. Express C in terms of A and B, using only the basic operations "union," "intersection," and "complement."

2.11 ⊡ An experiment has only two outcomes. The first has probability p to occur, the second probability p^2. What is p?

2.12 ⊞ In the UEFA Euro 2004 playoffs draw 10 national football teams were matched in pairs. A lot of people complained that "the draw was not fair," because each strong team had been matched with a weak team (this is commercially the most interesting). It was claimed that such a matching is extremely unlikely. We will compute the probability of this "dream draw" in this exercise. In the spirit of the three-envelope example of Section 2.1 we put the names of the 5 strong teams in envelopes labeled $1, 2, 3, 4$, and 5 and of the 5 weak teams in envelopes labeled $6, 7, 8, 9$, and 10. We shuffle the 10 envelopes and then match the envelope on top with the next envelope, the third envelope with the fourth envelope, and so on. One particular way a "dream draw" occurs is when the five envelopes labeled $1, 2, 3, 4, 5$ are in the odd numbered positions (in any order!) and the others are in the even numbered positions. This way corresponds to the situation where the first match of each strong team is a home match. Since for each pair there are two possibilities for the home match, the total number of possibilities for the "dream draw" is $2^5 = 32$ times as large.

- **a.** An outcome of this experiment is a sequence like $4, 9, 3, 7, 5, 10, 1, 8, 2, 6$ of labels of envelopes. What is the probability of an outcome?
- **b.** How many outcomes are there in the event "the five envelopes labeled $1, 2, 3, 4, 5$ are in the odd positions—in any order, and the envelopes labeled $6, 7, 8, 9, 10$ are in the even positions—in any order"?
- **c.** What is the probability of a "dream draw"?

2.13 In some experiment first an arbitrary choice is made out of four possibilities, and then an arbitrary choice is made out of the remaining three possibilities. One way to describe this is with a product of two sample spaces $\{a, b, c, d\}$:

$$\Omega = \{a, b, c, d\} \times \{a, b, c, d\}.$$

- **a.** Make a 4×4 table in which you write the probabilities of the outcomes.
- **b.** Describe the event "c is one of the chosen possibilities" and determine its probability.

2.14 ⊞ Consider the Monty Hall "experiment" described in Section 1.3. The door behind which the car is parked we label a, the other two b and c. As the sample space we choose a product space

$$\Omega = \{a, b, c\} \times \{a, b, c\}.$$

Here the first entry gives the choice of the candidate, and the second entry the choice of the quizmaster.

a. Make a 3×3 table in which you write the probabilities of the outcomes. *N.B.* You should realize that the candidate does *not know* that the car is in a, but the quizmaster will never open the door labeled a because he *knows* that the car is there. You may assume that the quizmaster makes an arbitrary choice between the doors labeled b and c, when the candidate chooses door a.

b. Consider the situation of a "no switching" candidate who will stick to his or her choice. What is the event "the candidate wins the car," and what is its probability?

c. Consider the situation of a "switching" candidate who will not stick to her choice. What is now the event "the candidate wins the car," and what is its probability?

2.15 The rule $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ from Section 2.3 is often useful to compute the probability of the union of two events. What would be the corresponding rule for three events A, B, and C? It should start with

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \cdots .$$

Hint: you could use the sum rule suitably, or you could make a diagram as in Figure 2.1.

2.16 ⊞ Three events E, F, and G cannot occur simultaneously. Further it is known that $\mathrm{P}(E \cap F) = \mathrm{P}(F \cap G) = \mathrm{P}(E \cap G) = 1/3$. Can you determine $\mathrm{P}(E)$?
Hint: if you try to use the formula of Exercise 2.15 then it seems that you do not have enough information; make a diagram instead.

2.17 A post office has two counters where customers can buy stamps, etc. If you are interested in the number of customers in the two queues that will form for the counters, what would you take as sample space?

2.18 In a laboratory, two experiments are repeated every day of the week in different rooms until at least one is successful, the probability of success being p for each experiment. Supposing that the experiments in different rooms and on different days are performed independently of each other, what is the probability that the laboratory scores its first successful experiment on day n?

2.19 ⊡ We repeatedly toss a coin. A head has probability p, and a tail probability $1 - p$ to occur, where $0 < p < 1$. The outcome of the experiment we are interested in is the number of tosses it takes until a head occurs for the *second* time.

a. What would you choose as the sample space?

b. What is the probability that it takes 5 tosses?